Fundamentals of Boundary-Layer Meteorology

Solutions Manual

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Contents

Chapter 1

Overview of the Solution Manual

(Written by Xuhui Lee)

Background

This solutions manual is the collective effort of participants in the *Boundary-Layer Meteorology* class taught in the School of Forestry and Environmental Studies, Yale University, in the fall of 2017. Each student was assigned 4-5 problems each week, and their solutions were submitted to a shared platform accessible by the whole class and the instructor for comments and corrections. Each problem was worked on by at least two students, and the best solution was then chosen for the compilation. In some cases, multiple solutions for the same problem are included in the manual so the user can benefit from the rich insights generated by their diverse methodology. In the last week of the class, the students were tasked with checking all the answers for correctness and for proper English use. The final edition has incorporated their corrections and corrections made by the instructor.

The same mathematical symbols in the textbook are used throughout the solutions manual. Equations are typeset following a similar format used by the textbook. Equations and other display items in the text are referred to as "Original Equation 3.2", "Original Table 5.7", and so on. For the reader's convenience, if an equation from the text is used for a solution, it is reproduced here and with the original equation number noted. Except for typographic errors, language editing was kept to a minimal level. By preserving the original "student voice", we hope that the manual is easily accessible to people new to the field of boundary-layer meteorology.

Some problems require numerical approximation to integration and differentiation or numerical solution of differential equations. The program code (Matlab or R) is attached at the end of the answer.

Purpose

The primary goal of compiling this solutions manual is to aid instructors and their assistants with classroom teaching. The answer keys and the detailed steps on how to reach the answer may be helpful to them when evaluating assignments done by their own students.

This solutions manual is also a supplementary teaching resource. Some of the problems can be used for expanded discussion and lectures on practical applications, serving to to balance the text's heavy emphasis on fundamental concepts and theories. A good example concerns the question about how to minimize flux measurement errors. The relevant exercises on this are found in three chapters (Problems 3.14, 3.16, 8.6, 8.7, 8.19, 9.7, 9.8, 9.9, 9.11 and 9.17). A focused session on these problems will be a good preparation if some of your students will be engaged in micrometeorological field experiments.

The second example is the relationship between the energy balance principle and the urban heat island phenomenon. Here, we approach the matter from several different angles with Problems 10.4, 10.11, 10.14, 10.15 and 10.16. The same principle can be extended to studies of temperature perturbations caused by other land use activities. The phenomenon of "green oasis" is mentioned in Problem 10.11. Using the energy redistribution factors given by Problems 10.15 and 10.16, it is not difficult to estimate the oasis effect, or the temperature difference between an irrigated farmland and its surrounding dry land, as long as the Bowen ratio and albedo contrasts between the two land types are known.

The third practical issue is related to air quality prediction. In Problem 7.6, students are asked to calculate the ground-level concentration of sulfur dioxide at several discrete points downwind of a smokestack. The same model can be used to map out the two-dimensional concentration field under various air stability and wind speed conditions and for different stack heights. Now let us suppose that someone proposes to build a power plant at a specific location near your university. By overlaying the concentration fields on a local map, students will find out whose neighborhood will be most impacted by this hypothetical power plant.

Students who have strong background in mathematics and are motivated by the topic may welcome the challenge provided by some of the difficult problems (marked with an asterisk). These problems can serve as launching points for small research projects. For example, Problem 6.9 is an attempt to improve the original Ekman solution of wind profiles in the boundary layer. The results indicate a sudden change in the wind speed across the top of the boundary layer, similar to the inversion jumps of scalar quantities in the slab ABL model (Chapter 11). It seems that the degree of wind speed discontinuity is related to surface roughness (Figures 6.3 and 6.4). This surface roughness connection can be further investigated with the program code supplied for the problem. Another aspect worth pursuing is sensitivity of the boundary-layer wind profile to eddy diffusivity parameterization. An additional benefit brought by research projects of this kind is increased competence in the numerical solutions of differential equations.

Another potential class project can be built on the slab model discussed in Chapter 11. In Problems 11.10 and 11.21, the ABL budgets of heat and carbon dioxide are quantified with synthetic time series of surface heat and carbon dioxide fluxes. More realistic results can be obtained by replacing these synthetic fluxes with actual observations. The project may yield exciting insights regarding mechanisms that govern the entrainment carbon dioxide flux, a subject that is poorly understood at the moment.

Future Updates

We plan to update this solutions manual periodically. Please send your error corrections to xuhui.lee@yale.edu. If you or your students have found a better solution and would like to share it with other users, we will be happy to include your contribution in the next edition of this manual.

Chapter 2

Fundamental Equations

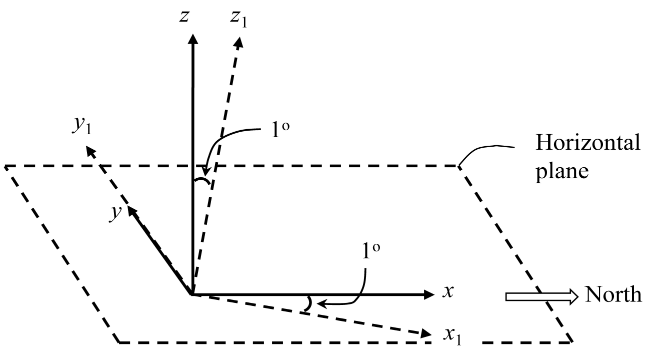

Figure 2.1: (Original Fig. 2.5) A tilted instrument coordinate $\{x_1, y_1, z_1\}$.

2.1* A sonic anemometer is mounted with its orientation toward N and at a downward tilt angle of $1°$ from the horizontal (Figure 2.1). The instrument expresses the velocity in a right-handed Cartesian coordinate $\{x_1, y_1, z_1\}$. Assume that the true air velocity is 5.00 $\mathrm{m\,s^{-1}}$ and the velocity vector lies perfectly in a horizontal plane. Determine the vertical velocity w_1 in the z_1 direction as a function of wind direction. This would be the vertical velocity measured by the instrument. Note that wind direction is $0°$ if wind blows from N, $90°$ if from E, and so on.

First, calculate u component of wind vector using wind speed and direction:

$$u = -U\cos(\theta) \tag{2.1}$$

where U is wind speed and θ is wind direction ($0°$ is North, $90°$ East, etc., Figure 2.2). Keep in mind that u is positive for south wind and negative for north wind according to Fig. 2.1.

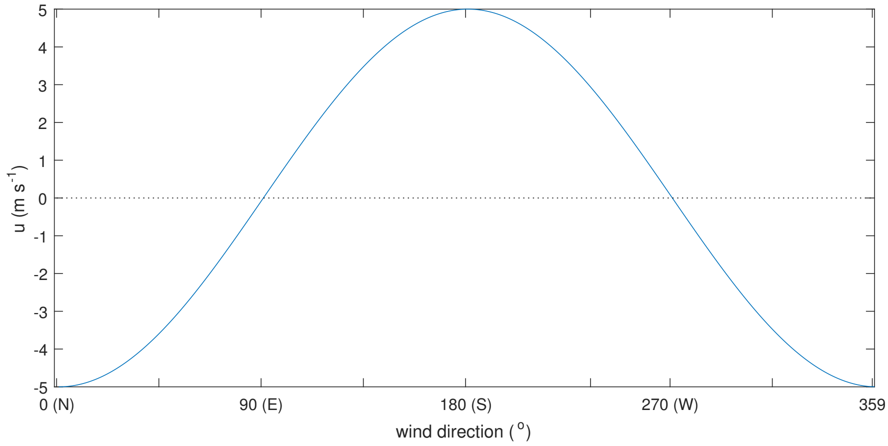

Figure 2.2: Wind speed in the x direction

Once we have the u wind component, we can calculate w_1 using:

$$\begin{aligned} w_1 &= w\cos(\alpha) + u\sin(\alpha) \\ &\simeq w + u\sin(\alpha) \\ &= u\sin(\alpha) \end{aligned} \tag{2.2}$$

where α is the 1^o tilt of the sonic anemometer in the North direction. We have used the fact that the velocity vector lies in the x-y plane so that $w = 0$. The vertical velocity w_1 in the z_1 direction (1^o tilt) as a function of wind direction is displayed in Figure 2.3. The tilt error is largest when wind direction is from the North or South, with w_1 reaching a magnitude of $0.087\ \mathrm{m\,s^{-1}}$.

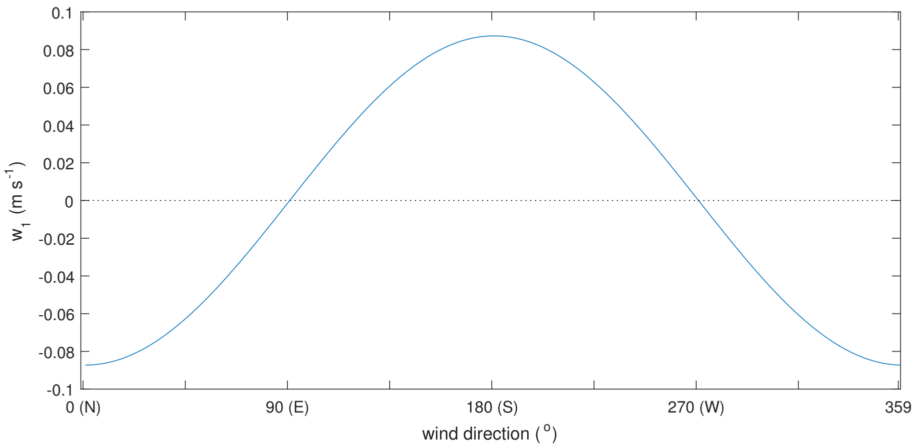

Figure 2.3: The instrument w_1 wind speed

2.2 Show that the viscous terms in Original Equations 2.3, 2.4 and 2.5 have the dimensions of acceleration.

The viscous forces in the x-direction, y-direction, and z-direction are given by $\nu\nabla^2 u$, $\nu\nabla^2 v$, and $\nu\nabla^2 w$, respectively.

Here, ν is the kinematic viscosity and has the unit of m^2 s^{-1}.

∇^2 is the Laplace operator $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Its unit is m^{-2}.

Finally, w, v, and w have the units of m s^{-1}.

Putting them all together, the unit of viscous force is m^2 s^{-1} $\cdot$ m^{-2} $\cdot$ m s^{-1} $\Rightarrow$ m s^{-2}.

Thus, the dimension of viscous force is L T^{-2}, the same as that of acceleration.

2.3 The mean velocity profile in the surface layer under neutral stability is described by the logarithmic function

$$\overline{u} = \frac{u_*}{k} \ln\left(\frac{z}{z_0}\right), \tag{2.3}$$

where k $(= 0.4)$ is the von Karman constant, u_* is friction velocity, z is height and z_0 is momentum roughness. Assume that the mean lateral and vertical velocity components are zero, and $u_* = 0.52$ m s^{-1}. Calculate the viscous force for two heights $z = 0.05$ m and 10 m. Use a kinematic viscosity value of $\nu = 1.48 \times 10^{-5}$ m^{2}s^{-1} at temperature 15°C for your calculations.

Given:

Von Karman constant $(k) = 0.4$

Friction velocity $(u_*) = 0.52$ m s^{-1}

Kinematic viscosity $(\nu) = 1.48 \times 10^{-5}$ m^2 s^{-1}

Since mean lateral and vertical velocity components are zero, there is only viscous force in the x-direction and the mean velocity profile is equal to u, the x-direction component of velocity.

The mean velocity is given by:

$$\bar{u} = \frac{u_*}{k} \ln\left(\frac{z}{z_0}\right)$$

The viscous force in the x-direction is:

$$\nu\nabla^2 u = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Putting $u = \bar{u} = \frac{u_*}{k}\ln\left(\frac{z}{z_0}\right)$, the partial differentials with respect to x and y become zero. Thus, the viscous force:

$$\begin{aligned}
&\nu\frac{\partial^2}{\partial z^2}(\frac{u_*}{k}\ln\left(\frac{z}{z_0}\right))\\
\Rightarrow\ &\nu\frac{u_*}{k}\frac{\partial^2}{\partial z^2}(\ln\left(\frac{z}{z_0}\right))\\
\Rightarrow\ &-\nu\frac{u_*}{k}\frac{1}{z^2}
\end{aligned}$$

For $z = 0.05\,\text{m}$:

$$\begin{aligned}
u &= -\nu\frac{u_*}{k}\frac{1}{z^2}\\
&= -1.48\times 10^{-5}\text{m}^2\ \text{s}^{-1}\times\frac{0.52\ \text{m s}^{-1}}{0.4}\frac{1}{(0.05\ \text{m})^2}\\
&\approx -7.7\times 10^{-3}\text{m s}^{-2}
\end{aligned}$$

For $z = 10\,\text{m}$:

$$\begin{aligned}
u &= -\nu\frac{u_*}{k}\frac{1}{z^2}\\
&= -1.48\times 10^{-5}\text{m}^2\ \text{s}^{-1}\times\frac{0.52\ \text{m s}^{-1}}{0.4}\frac{1}{(10\ \text{m})^2}\\
&\approx -1.9\times 10^{-7}\text{m}\,\text{s}^{-2}
\end{aligned}$$

The viscous force is much smaller than the pressure gradient force except very close to the surface and therefore can be omitted.

2.4 Derive the conservation equation for the CO_2 mass mixing ratio (Original Equation 2.18) from Original Equations 2.15 and 2.17.

First for mixing ratio s_c we have:

$$\begin{aligned}
\frac{ds_c}{dt} &= \frac{d}{dt}\frac{\rho_c}{\rho_d} = \frac{1}{\rho_d}\frac{d\rho_c}{dt} - \frac{\rho_c}{\rho_d^2}\frac{d\rho_d}{dt}\\
\frac{1}{\rho_d}\frac{d\rho_c}{dt} &= \frac{ds_c}{dt} + \frac{\rho_c}{\rho_d^2}\frac{d\rho_d}{dt}
\end{aligned}$$

Divide the carbon dioxide for mass continuity equation (Original Equation 2.17) by ρ_d:

$$\begin{aligned}
\frac{1}{\rho_d}\frac{d\rho_c}{dt} + \frac{\rho_c}{\rho_d}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) &= \frac{s_c}{\rho_d} + \frac{\kappa_c}{\rho_d}\nabla^2\rho_c\\
\frac{ds_c}{dt} + \frac{\rho_c}{\rho_d^2}\frac{d\rho_d}{dt} + \frac{\rho_c}{\rho_d}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) &= \frac{s_c}{\rho_d} + \frac{\kappa_c}{\rho_d}\nabla^2\rho_c
\end{aligned}$$

However, dry air mass continuity equation requires that:

$$\frac{\rho_c}{\rho_d^2}\frac{d\rho_d}{dt} + \frac{\rho_c}{\rho_d}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = \frac{\rho_c}{\rho_d}[\frac{1}{\rho_d}\frac{d\rho_d}{dt} + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})] = 0$$

Therefore, we get:

$$\frac{ds_c}{dt} = \frac{s_c}{\rho_d} + \kappa_c \nabla^2 s_c.$$

2.5 Using the ideal gas law relations and the continuity equation, derive the energy conservation equation (Original Equation 2.23) from Original Equation 2.22.

From ideal gas law for dry air:

$$p_d = \rho_d R_d T$$

we have:

$$\frac{dp_d}{dt} = \rho_d R_d \frac{dT}{dt} + R_d T \frac{d\rho_d}{dt}$$

Substitute the above equation into Original Equation 2.22 and use Original Equation 2.33:

$$\rho_d c_v \frac{dT}{dt} + \rho_d R_d \frac{dT}{dt} = \rho_d R_d \frac{dT}{dt} + R_d T \frac{d\rho_d}{dt} + \rho_d c_p S_T + \rho_d p_p \kappa_T \nabla^2 T$$

From the mass continuity equation and the ideal gas law:

$$\begin{aligned} R_d T \frac{d\rho_d}{dt} &= -R_d T \rho_d (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \\ &= -p_d (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \\ &\simeq -p (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \end{aligned}$$

we reach:

$$\rho_d c_v \frac{dT}{dt} = -p(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \rho_d c_p S_T + \rho_d c_p \kappa_T \nabla^2 T.$$

2.6 Derive the potential temperature conservation equation (Original Equation 2.25) from the energy conservation equation (Original Equation 2.22).

Potential temperature is given by:

$$\theta = T(\frac{p}{p_0})^{-\frac{R_d}{c_p}}$$

Taking total time derivative:

$$
\begin{aligned}
\frac{d\theta}{dt} &= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{dT}{dt} + T\frac{d}{dt}\left[\frac{1}{(\frac{p}{p_0})^{\frac{R_d}{c_p}}}\right] \\
&= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{dT}{dt} + T\frac{-R_d}{c_p}(\frac{p}{p_0})^{\frac{-R_d}{c_p}-1}\frac{d}{dt}(\frac{p}{p_0}) \\
&= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{dT}{dt} + T\frac{-R_d}{c_p}(\frac{p}{p_0})^{\frac{-R_d}{c_p}-1}\frac{1}{p_0}\frac{p}{p}\frac{dp}{dt} \\
&= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{dT}{dt} - T\frac{R_d}{c_p p}(\frac{p}{p_0})^{\frac{-R_d}{c_p}}\frac{dp}{dt} \\
&= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{dT}{dt} - \frac{\cancel{p}}{c_p\rho_d\cancel{p}}(\frac{p}{p_0})^{\frac{-R_d}{c_p}}\frac{dp}{dt} \quad \text{(from the ideal gas law (Original Equation 2.27))} \\
&= (\frac{p}{p_0})^{-\frac{R_d}{c_p}}\frac{1}{\rho_d c_p}\left[\frac{dp}{dt} + \rho_d c_p S_T + \rho_d c_p \kappa_T \nabla^2 T\right] - \frac{1}{c_p\rho_d}(\frac{p}{p_0})^{\frac{-R_d}{c_p}}\frac{dp}{dt}
\end{aligned}
$$

(from the principle of energy conservation (Original Equation 2.22))

$$
\begin{aligned}
&= S_T(\frac{p}{p_0})^{-\frac{R_d}{c_p}} + \kappa_T\nabla^2 T(\frac{p}{p_0})^{-\frac{R_d}{c_p}} + \cancelto{0}{\left[\frac{1}{c_p\rho_d}(\frac{p}{p_0})^{\frac{-R_d}{c_p}}\frac{dp}{dt} - \frac{1}{c_p\rho_d}(\frac{p}{p_0})^{\frac{-R_d}{c_p}}\frac{dp}{dt}\right]} \\
&= S_\theta + \kappa\nabla^2\theta
\end{aligned}
$$

where $S_\theta = S_T(\frac{p}{p_0})^{-\frac{R_d}{c_p}}$

In the above derivation, we have approximated the dry air pressure p_d by the total pressure p. To avoid this approximation, ρ_d (dry air mass density) in Original Equation 2.22 should be replaced by ρ (moist air mass density), and the potential temperature should be defined more rigorously as:

$$\theta = T(\frac{p}{p_0})^{-R_m/c_p}$$

where R_m is the ideal gas constant for moist air (J. M. Wallace and P. V. Hobbs, 1977, *Atmospheric Science: An Introductory Survey*, Academic Press).

2.7 Which of the following quantities of the air parcel are conserved during the dry adiabatic process shown in Original Figure 2.3: air pressure, dry air density, air temperature, potential temperature, water vapor density, water vapor partial pressure, water vapor mixing ratio, carbon dioxide density, carbon dioxide partial pressure, and carbon dioxide mixing ratio? Is the incompressibility condition satisfied? (Note that no phase change of water occurs in the dry adiabatic process.)

During the dry adiabatic process shown in Original Figure 2.3, potential temperature, water vapor mixing ratio, and carbon dioxide mixing ratio are conserved. This can be seen from their respective conservation equations (Original Equations 2.25, 2.20, 2.18), keeping in mind that during the dry adiabatic process, there is no molecular diffusion between the air parcel and the surrounding air and no internal sources of heat, water vapor, and carbon dioxide exists in the air parcel.

The temperature of the air parcel will increase with decreasing height. The mass densities are not conserved. Neither are the partial and total pressures.

The incompressibility condition is not satisfied.

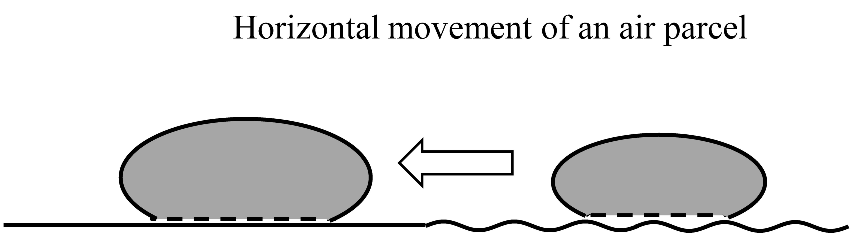

Figure 2.4: (Original Figure 2.6) Movement of an air parcel from a cool lake to a hot cement parking lot.

2.8 An air parcel moves horizontally from a lake surface to a hot paved parking lot (Figure 2.4). The situation is diabatic because the parcel can now exchange materials and energy with the surface through its bottom boundary which is in contact with the surface. Which of the quantities listed in Problem 2.7 are conserved? Is the incompressibility condition satisfied?

In this case, the air parcel will exchange heat and materials with the underlying surface. The only quantity that stays unchanged over time may be its carbon dioxide mixing ratio, assuming that the parking lot and the lake neither absorb carbon dioxide from or release carbon dioxide to the air parcel.

The incompressibility condition is not satisfied because the dry air mass density is not conserved.

2.9 Show that the diffusion term of the energy conservation equation (third term on the right side of Original Equation 2.22) has the same dimensions as the internal energy change term (term on the left) and the volume expansion term (first term on the right). In what dimensions should the source term S_T be?

The equation is $\rho_d c_p \frac{dT}{dt} = \frac{dp}{dt} + \rho_d c_p S_T + \rho_d c_p \kappa_T \nabla^2 T$

Diffusion term: $\rho_d c_p \kappa_T \nabla^2 T$ has units of (kg m^{-3})(J kg^{-1} K^{-1})(m^2 s^{-1})(K m^{-2}), which simplifies to kg m^{-1} s^{-3}

Internal energy change term: $\rho_d c_p \frac{dT}{dt}$ has units of (kg m^{-3})(J kg $^{-1}$ K^{-1})(K^{-1} s^{-1}), which simplifies to kg m^{-1} s^{-3}

Volume expansion term: $\frac{dp}{dt}$ has units of Pa s^{-1}, which is the same as $\mathrm{N\,m^{-2}\,s^{-1}}$, or kg m^{-1} s^{-3}, remembering that $\mathrm{N} = \mathrm{kg\,m\,s^{-2}}$.

So all three terms have the same dimensions.

The source term S_T must have dimensions of $\mathrm{K\,s^{-1}}$ in order for $\rho_d c_p S_T$ to have the same dimensions as the other terms of the conservation equation.

2.10 The atmospheric pressure is 1000.2 hPa, the water vapor molar mixing ratio is 19.27 mmol mol^{-1}, and the carbon dioxide molar mixing ratio is 400.4 μmol mol^{-1}. Determine the partial pressures of dry air, water vapor, and carbon dioxide. What is the mass density of dry air, water vapor and carbon dioxide if the air temperature is 15.0°C?

Water vapor molar mixing ratio is $\chi_v = \frac{p_v}{p_d} = 19.27$ mmol mol^{-1}.

We obtain the partial pressure of water vapor $p_v = 19.27$ mmol mol$^{-1}p_d$.

Dalton's law of partial pressures states that total atmospheric pressure is approximately the sum of the partial pressures of dry air and water vapor:

$p = p_d + p_v = 1000.2\text{hPa} = p_d + (19.27 \text{ mmol mol}^{-1} p_d)$

Partial pressure of dry air: $p_d = \frac{1000.2\text{hPa}}{1+19.27 \text{ mmol/mol}} = 981.3$ hPa

Partial pressure of water vapor: $p_v = p - p_d = 1000.2$ hPa -981.3 hPa $= 18.9$ hPa

Carbon dioxide molar mixing ratio is $\chi_c = \frac{p_c}{p_d} = 400.4\mu\text{mol mol}^{-1}$

Partial pressure of carbon dioxide: $p_c = \chi_c p_d = (400.4\mu\text{mol mol}^{-1})$ (981.3 hPa) $= 0.3929$ hPa

The ideal gas law for dry air is $p_d = \rho_d R_d T$

R_d is the ideal gas constant for dry air, and $R_d = \frac{R}{M_d} = \frac{8.314 \text{ J mol}^{-1} \text{ K}^{-1}}{0.029 \text{ kg mol}^{-1}} = 286.7$ J kg^{-1} K^{-1}

The ideal gas law for water vapor is $p_v = \rho_v R_v T$

R_v is the ideal gas constant for water vapor, and $R_v = \frac{R}{M_v} = \frac{8.314 \text{ J mol}^{-1} \text{ K}^{-1}}{0.018 \text{ kg mol}^{-1}} = 461.9$ J kg^{-1} K^{-1}

The ideal gas law for carbon dioxide is $p_c = \rho_c R_c T$

R_c is the ideal gas constant for carbon dioxide, and $R_c = \frac{R}{M_c} = \frac{8.314 \text{ J mol}^{-1} \text{ K}^{-1}}{0.044 \text{ kg mol}^{-1}} = 189.0$ J kg^{-1} K^{-1}

If the air temperature is 15°C, or 288.15 K:

Rearranging the ideal gas law for dry air we get $\rho_d = \frac{p_d}{R_d T} = \frac{981.3 \text{ hPa}}{(286.7 \text{ J kg}^{-1} \text{K}^{-1})(288.15 \text{ K})}$

The mass density of dry air: $\rho_d = 1.19$ $\mathrm{kg\,m^{-3}}$

Rearranging the ideal gas law for water vapor we get $\rho_v = \frac{p_v}{R_v T} = \frac{18.9 \text{ hPa}}{(461.9 \text{ J kg}^{-1} \text{K}^{-1})(288.15 \text{ K})}$

The mass density of water vapor: $\rho_v = 0.0142$ kg m^{-3} or 14.2 g m^{-3}

Rearranging the ideal gas law for carbon dioxide we get $\rho_c = \frac{p_c}{R_c T} = \frac{0.3929 \text{ hPa}}{(189.0 \text{ J kg}^{-1}\text{ K}^{-1})(288.15 \text{ K})}$

The mass density of carbon dioxide: $\rho_c = 7.21$ x 10^{-4} kg m^{-3} or 721 mg m^{-3}

2.11 CO_2 is a well-mixed gas in the lower atmosphere, meaning that its long-term mean mixing ratio does not vary with height. The global mean CO_2 molar mixing ratio was 396.5 μmol mol^{-1} in 2013. Estimate the vertical CO_2 mass density gradient in the atmospheric boundary layer of the standard atmosphere. (Hints: In the standard atmosphere, the pressure and the temperature are 1013.2 hPa and 15.0°C at the sea level and 898.7 hPa and 8.5°C at the altitude of 1000 m. Use a typical water vapor mixing ratio of 15 g kg^{-1} for both heights.)

Water vapor mass mixing ratio is:

$$s_v = \frac{M_v}{M_d}\frac{p_v}{p_d}$$

At sea level, this equation along with Dalton's law of partial pressure gives us:

$$15 \text{ g kg}^{-1} = \frac{0.018 \text{ kg mol}^{-1}}{0.029 \text{ kg mol}^{-1}} \times \frac{1013.2 \text{ hPa} - p_d}{p_d}$$

Solving for partial pressure of dry air: $p_d = 989.3$ hPa

Carbon dioxide molar mixing ratio is:

$$\chi_c = \frac{p_c}{p_d}$$

Solving for partial pressure of carbon dioxide:

$$p_c = \chi_c p_d = (396 \ \mu\text{mol mol}^{-1})(989.3 \text{ hPa}) = 0.392 \text{ hPa}$$

The ideal gas law for carbon dioxide is $p_c = \rho_c R_c T$

Solving for mass density of carbon dioxide at sea level:

$$\rho_c = \frac{p_c}{R_c T} = \frac{0.392 \text{ hPa}}{(189.0 \text{ J kg}^{-1}\text{ K}^{-1})(288.15 \text{ K })} = 7.19 \times 10^{-4} \text{ kg m}^{-3}$$

At a height of 1000 m:

$$15 \text{ g kg}^{-1} = \frac{0.018 \text{ kg mol}^{-1}}{0.029 \text{ kg mol}^{-1}} \times \frac{898.7 \text{ hPa} - p_d}{p_d}$$

Solving for partial pressure of dry air, $p_d = 877.5$ hPa

Carbon dioxide molar mixing ratio is:

$$\chi_c = \frac{p_c}{p_d}$$

Solving for partial pressure of carbon dioxide:

$$p_c = \chi_c p_d = (396 \ \mu\text{mol mol}^{-1})(877.5 \text{ hPa}) = 0.347 \text{ hPa}$$

The ideal gas law for carbon dioxide is $p_c = \rho_c R_c T$

Solving for mass density of carbon dioxide the height of 1000 m:

$$\rho_c = \frac{p_c}{R_c T} = \frac{0.347 \text{ hPa}}{(189.0 \text{ J kg}^{-1} \text{ K}^{-1})(281.65 \text{ K})} = 6.53 \times 10^{-4} \text{ kg m}^{-3}$$

The vertical carbon dioxide mass density gradient in the atmospheric boundary layer of the standard atmosphere is the difference of mass density between 1000 m and sea level divided by the height difference:

$$\frac{\rho_{c1} - \rho_{c2}}{z_1 - z_2} = \frac{7.19 \times 10^{-4} \text{ kg m}^{-3} - 6.53 \times 10^{-4} \text{ kg m}^{-3}}{1000 \text{ m}}$$
$$= 6.60 \times 10^{-8} \text{ kg m}^{-4}$$

2.12 An air parcel is adiabatically lifted upward from the sea level. Its initial temperature is 20.0°C, its water vapor pressure is 6.22 hPa, and its CO_2 mass density is 800.0 mg m^{-3}. What will be its temperature and CO_2 mass density when the parcel reaches the pressure height of 700 hPa?

With the definition of potential temperature in an adiabatic process:

$$\theta = T \, (\frac{p}{p_0})^{-\frac{R_d}{c_p}}$$

where p_0 is pressure at the mean sea level (1013.25 hPa) and R_d is the ideal gas constant for dry air (8.314 J mol^{-1}K^{-1}). For air, $R_d/c_p \approx 0.286$. Here θ is 293 K and p is 700 hPa. Thus:

$$T = 263.6K = -9.4°\text{C}$$

According to the ideal gas law, the following formula exists for CO_2:

$$p_c = \rho_c R_c T$$

For a unit volume (m^3) of air, the mass of CO_2 stays constant.

Noting the ideal gas law expression:

$$\frac{p_0 V_0}{T_0} = \frac{pV}{T}$$

So the volume after being lifted is 1.30 m^3. Thus, the mass density:

$$\rho_c = 800.0/1.30 = 615.4 \text{ mg m}^{-3}$$

An alternative approach is to use the fact that the mixing ratios of carbon dioxide and water vapor remain unchanged during the adiabatic accent. The molar mixing ratio of water vapor at sea level is:

$\chi_v = p_v/(p - p_v) = 6.22/(1013.25\text{-}6.22) = 6.18 \text{ mmol mol}^{-1}$.

At the upper level, the molar mixing ratio of water vapor is the same, but the vapor partial pressure has changed. From $p_v/(700 - p_v) = 6.18 \text{ mmol mol}^{-1}$, we obtain:

$p_v = 4.30$ hPa

The dry air density of CO_2 at sea level is:

$\rho_d = \mathrm{p}_d/(R_d T) = (1013.25 - 6.22)/(287\times\ 293) = 1.198 \text{ kg m}^{-3}$

This gives the mass mixing ratio of CO_2:

$s_c = 800.0 \text{ mg m}^{-3}/\ (1.198 \text{ kg m}^{-3}) = 667.8 \text{ mg kg}^{-1}$

From Original Equation 2.34, we obtain the molar mixing ratio of CO_2:

$\chi_c = 440.1\ \mu\text{mol mol}^{-1}$

The partial pressure of CO_2 at the upper level is:

$p_c = (440.1\ \mu\text{mol mol}^{-1}) \times (700 \text{ hPa} - 4.30 \text{ hPa}) = 30.6$ Pa

The mass density of CO_2 at the upper level is obtained from the ideal gas law:

$\rho_d = p_c/(R_c T) = 30.6 \text{ Pa} /(189 \text{ J kg}^{-1}\text{ K}^{-1}\ 263.6 \text{ K}) = 6.142 \times 10^{-4} \text{ kg m}^{-3} = 614.2 \text{ mg m}^{-3}$

2.13 The annual mean carbon dioxide flux of a forest is -0.37 $\text{mg CO}_2\text{ m}^{-2}\text{ s}^{-1}$. Express the flux in the dimensions of $\mu\text{mol m}^{-2}\text{ s}^{-1}$, $\text{g C m}^{-2}\text{y}^{-1}$, $\text{g CO}_2\text{ m}^{-2}\text{y}^{-1}$ and $\text{t C ha}^{-1}\text{y}^{-1}$.

- $-0.37 \text{ mg CO}_2\text{ m}^{-2}\text{ s}^{-1}$
- $(-0.37 \text{ mg CO}_2\text{ m}^{-2}\text{ s}^{-1}\ ) \times (1 \text{ mol} / 44.01 \text{ g}) \times (1 \text{ g} / 1000 \text{ mg}) \times (1{,}000{,}000\ \mu\text{mol}/ 1 \text{ mol}) = -8.4\ \mu \text{ mol m}^{-2}\text{ s}^{-1}$
- $(-8.4\ \mu\text{mol m}^{-2}\text{ s}^{-1}) \times (12.011 \text{ g C} / 1 \text{ mol}) \times (31{,}536{,}000 \text{ s} / 1 \text{ yr}) \times (1 \text{ mol} / 1{,}000{,}000\ \mu\text{mol}) = -3200 \text{ g C m}^{-2}\text{ yr}^{-1}$
- $(-8.4\ \mu\text{mol m}^{-2}\text{ s}^{-1}) \times (44.01 \text{ g CO}_2 / 1 \text{ mol}) \times (31{,}536{,}000 \text{ s} / 1 \text{ yr}) \times (1 \text{ mol} / 1{,}000{,}000\ \mu\text{mol}) = -12,000 \text{ g CO}_2\text{ m}^{-2}\text{ yr}^{-1}$
- $(-3200 \text{ g C m}^2\text{ yr}^{-1}) \times (\ 1 \text{ t} / 1{,}000{,}000 \text{ g}) \times (10{,}000 \text{ m}^2 / 1 \text{ ha}) = -32 \text{ t C ha}^{-1}\text{ yr}^{-1}$

This calculation is purely hypothetical. No natural ecosystem can remove this amount of atmospheric carbon dioxide annually.

2.14 A typical CH_4 flux in a mid-latitude wetland is 200 $\mathrm{nmol\,m^{-2}\,s^{-1}}$ in the warm season. Express the flux in the dimensions of $\mu\mathrm{g\,CH_4\,m^{-2}\,s^{-1}}$ and $\mathrm{mg\,CH_4\,m^{-2}\,d^{-1}}$.

- 200 $\mathrm{nmol\,m^{-2}\,s^{-1}}$
- (200 $\mathrm{nmol\,m^{-2}\,s^{-1}}$) $\times$ (1 mol / 1,000,000,000 nmol) $\times$ (16.04 g CH_4/ 1 mol) $\times$ (1,000,000 μg / 1 g) = 3 $\mu\mathrm{g\,CH_4\,m^{-2}\,s^{-1}}$
- (200 $\mathrm{nmol\,m^{-2}\,s^{-1}}$) $\times$ (1 mol / 1,000,000,000 nmol) $\times$ (16.04 g CH_4/ 1 mol) $\times$ (1,000 mg / 1 g) $\times$ (86,400 s / 1 d) = 300 $\mathrm{mg\,CH_4\,m^{-2}\,d^{-1}}$

2.15 The daily mean water vapor flux is 0.074 $\mathrm{g\,m^{-2}\,s^{-1}}$. How much water, in mm of water depth, is lost via evaporation in one day?

Assuming the density of liquid water (ρ_w) as 1000 $\mathrm{kg\,m^{-3}}$, we can convert the mass of evaporated water vapor (m_w) into the depth of liquid water (d_w) by calculating the volume of liquid water (V_w):

$$d_w = \frac{V_w}{A} = \frac{m_w}{\rho_w A}$$

where A is area.

For per square meter area, in 1 second, the water loss is 0.074 g. For the same area, in 1 day, the mass of water is:

$$0.074\ \mathrm{g\,s^{-1}} \times\ 24\ \mathrm{h\,d^{-1}} \times 60\ \mathrm{min\,h^{-1}} \times\ 60\ \mathrm{s\,min^{-1}} = 6.4\ \mathrm{kg\,d^{-1}}$$

Thus, the depth is:

$$d_d = \frac{6.4\ \mathrm{kg\,d^{-1}}}{1000\ \mathrm{kg\,m^{-3}} \times 1\ \mathrm{m^2}} = 6.4 \times 10^{-3}\ \mathrm{m} = 6.4\ \mathrm{mm\,d^{-1}}$$

So 6.4 mm water is lost via evaporation in 1 day.

2.16 One estimate of the solar radiation incident on the Earth's surface is 175 $\mathrm{W\,m^{-2}}$, the global means surface albedo is 0.126, and the mean incoming and outgoing longwave radiation are 344 and 396 $\mathrm{W\,m^{-2}}$, respectively (Zhao et al., 2013). Determine the global mean surface net radiation. The global mean annual precipitation is 1030 mm. If this precipitation water flux is balanced exactly by the surface evaporation rate, what are the global mean surface latent heat flux and sensible heat flux?

The global mean surface net radiation is determined by subtracting outgoing radiation from incoming radiation.

Incoming radiation consists of longwave and shortwave components. The later is the portion of solar radiation that is not reflected back to space. The portion is determined by $(1 - albedo)$:

$$K_{\downarrow} = 175 \times \ (1 - 0.126) = 153 \text{ W m}^{-2}$$

The longwave part:

$$L_{\downarrow} = 344 \text{ W m}^{-2}$$

The outgoing longwave radiation is:

$$L_{\uparrow} = 396 \text{ W m}^{-2}$$

Thus, the global mean surface net radiation:

$$R_{n,0} = 344 + 153 - 396 = 101 \text{ W m}^{-2}$$

Since the precipitation water flux is balanced by the surface evaporation rate, the water vapor flux is 1030 mm $\text{y}^{-1} = 0.033 \text{ g m}^{-2} \text{ y}^{-1}$. The latent heat of vaporization $\lambda = 2466 \text{ J g}^{-1}$. Thus, the latent heat flux is:

$$\lambda \text{E}_0 = 2466 \text{ J g}^{-1} \times 0.033 \text{ g m}^{-2} = 81 \text{ W m}^{-2}$$

Alternatively, we can obtain the global mean latent heat flux by unit conversion following these steps:

$$\lambda = 2466 \text{ J g}^{-1}$$

$$E_0 = 1030 \text{ mm yr}^{-1}$$

$$\lambda E_0 = 2{,}540{,}000 \text{ J mm g}^{-1} \text{yr}^{-}1 \times (1 \text{ yr } / \ 31{,}536{,}000 \text{ s}) \times (1 \text{ g } / \ 1000 \text{ mm}^3)$$

$$E_0 = 0.00008054 \text{ W mm}^{-2} \times (\ 1{,}000{,}000 \text{ mm}^2 \ / \ 1 \text{ m}^2) = 81 \text{ W m}^{-2}$$

According to the surface energy balance equation:

$$R_{n,0} = H_0 + \lambda E_0 + G_0$$

and if we suppose the ground flux to be negligible on the annual time scale, the sensible heat flux is

$$H_0 = R_{n,0} - \lambda E_0 = 20 \text{ W m}^{-2}$$

2.17 Bowen ratio, β, is the ratio of the surface sensible heat flux to the latent heat flux. A Bowen ratio apparatus determines β of a surface by measuring the vertical gradients of air temperature and humidity above the surface. The measured β is combined with simultaneous measurements of the available energy (net radiation $R_{n,0}$, soil heat flux G_0) to calculate the surface latent heat flux λE_0. On the basis of the energy balance principle, derive an expression that relates λE_0 to β, $R_{n,0}$ and G_0.

By definition:

$$\beta = \frac{H_0}{\lambda E_0} \tag{2.4}$$

In this scenario, measurements are made at the surface, so the surface energy balance can be described by the equation:

$$R_{n,0} = H_0 + \lambda E_0 + G_0 \tag{2.5}$$

Rearranging Eq.2.5, the two equations can be combined to yield the expression:

$$\lambda E_0 = \frac{R_{n,0} - G_0}{\beta + 1} \tag{2.6}$$

2.18 The incoming shortwave and longwave radiation are 750 and 419 $\mathrm{W\,m^{-2}}$, respectively, at a sub-tropical lake at noon on a summer day. The lake surface temperature is 25.3°C and the surface albedo is 0.06. Assuming that the lake surface is a black body, calculate the surface net radiation.

The surface net radiation, $R_{n,0}$, is a balance of incoming shortwave and longwave radiation, and the proportions of each that is reflected or emitted from the surface:

$$R_{n,0} = K_{\downarrow} - K_{\uparrow} + L_{\downarrow} - L_{\uparrow}$$

The shortwave radiation reflected from the surface is a function of the albedo and the incoming shortwave radiation:

$$K_{\uparrow} = \alpha K_{\downarrow}$$

The longwave radiation reflected and emitted from the surface is a function of the incoming longwave radiation, the emissivity, ϵ, the Stefan-Boltzmann constant, σ, and the surface temperature, T_s:

$$L_{\uparrow} = (1 - \epsilon)L_{\downarrow} + \epsilon\sigma T_s^4 \tag{2.7}$$

Since emissivity is equal to 1 in the case of a black body, the above equation simplifies to:

$$L_{\uparrow} = \epsilon\sigma T_s^4 \tag{2.8}$$

Thus, the radiation balance equation becomes:

$$R_{n,0} = K_{\downarrow} - \alpha K_{\downarrow} + L_{\downarrow} - \epsilon\sigma T_s^4 \tag{2.9}$$

After converting the temperature to Kelvin, the given values can simply be plugged into the above equation:

$R_{n,0}$= 750 W m^{-2} - 0.06 (750 W m^{-2}) + 419 W m^{-2} - (5.67 $\times$ 10^{-8} W m^{-2} K^{-4}) 298.45 K^4

Thus, the surface net radiation is 674 W m^{-2}.

2.19 The water vapor pressure is 12.2 hPa, the atmospheric pressure is 984.5 hPa, the air temperature is 17.6°C, and the molar mixing ratio of CO_2 is 409.7 μmol mol^{-1}. Find the CO_2 mass density in mg m^{-3}.

According to the ideal gas law for CO_2,

$$p_c = \rho_c R_c T$$

where p_c is the only variable unknown if we intend to get ρ_c.

Note that p_d is

$$p_d \approx p - p_v$$

and the molar mixing ratio of CO_2 (χ_c), which is known, can be expressed by:

$$\chi_c = \frac{p_c}{p_d}$$

Therefore, the p_c and p_d are given by the equation set:

$$p_d = 984.5 - 12.2 = 972.3 \text{ hPa}$$

$$\frac{p_c}{p_d} = 409.7 \ \mu\text{mol mol}^{-1}$$

The solution is:

$$p_c = 0.398 \text{ hPa}$$

Then the mass density of CO_2 is given by:

$$\rho_c = \frac{p_c}{R_c T}$$

The final answer: $\rho_c = 724.79$ mg m^{-3}

2.20 The air temperature is 15.9°, the atmospheric pressure is 998.3 hPa, and the water vapor mass density is 23.6 g m^{-3}. Find the vapor mass and molar mixing ratios.

Approach 1:

Similar to Question 2.19, this problem requires the use of the ideal gas law (Original Equation 2.27), Dalton's law of partial pressures (Original Equation 2.38) and molar mixing ratio (Original Equation 2.37), in addition to the relationship between mass mixing ratio and partial pressures:

$$s_v = \frac{M_v P_v}{M_d P_d} = \chi_v \frac{M_v}{M_d}$$

Combining Original Equations 2.27, 2.37, and 2.38 yields:

$$\chi_v = \frac{\rho_v R_v T}{P - \rho_v R_v T} \tag{2.10}$$

The molar mixing ratio of water pressure is, therefore, 32.6 $\text{mmol}\,\text{mmol}^{-1}$.

This value can then be plugged into Original Equation 2.34 to get a mass mixing ratio of 20.2 $\text{g}\,\text{kg}^{-1}$.

Approach 2:

The vapor mass and molar mixing ratios are given by:

$$s_v = \frac{M_v}{M_d}\frac{p_v}{p_d}$$

$$\chi_v = \frac{p_v}{p_d}$$

The mass mixing ratio s_v can be calculated if we have χ_v since:

$$s_v = \frac{M_v}{M_d}\chi_v$$

According to the ideal gas law for water vapor:

$$p_v = \rho_v R_v T = \rho_v \frac{R}{M_v} T$$

We obtain $p_v = 31.5$ hPa

Given $p \approx p_d + p_v$

So we get the dry air partial pressure $p_d = 966.8$ hPa.

Therefore, we have $\chi_v = 32.6$ mmol mol^{-1} and $s_v = 20.2$ g kg^{-1}.

2.21 The methane and nitrous oxide molar mixing ratios are 2.89 ppm and 401.2 ppb, respectively. Find their mass mixing ratios.

The given molar mixing ratios can simply be converted to mass mixing ratios using the relationship shown in Original Equations 2.34 and 2.36. For methane, the molar mass is $M = 0.016$ $\text{kg}\,\text{mol}^{-1}$ and for nitrous oxide, the molar mass is $M = 0.044$ $\text{kg}\,\text{mol}^{-1}$.

Thus, the molar mixing ratios for methane and nitrous oxide are 1.59 $\text{mg}\,\text{kg}^{-1}$ and 609 $\mu\text{g}\,\text{kg}^{-1}$, respectively.

Chapter 3

Governing Equations for Mean Quantities

3.1 Prove the first three Reynolds rules (Original Equations 3.3, 3.4, and 3.5).

Approach 1:

Original Equation 3.3:

$$\overline{a} = \frac{1}{n}\sum_{1}^{n} a_i$$

$$\overline{\overline{a}} = \frac{1}{n}\sum_{1}^{n} \overline{a_i} = \frac{1}{n}(n\overline{a})$$

$$\overline{\overline{a}} = \overline{a}$$

Original Equation 3.4:

$$a' = a - \overline{a}$$

$$\overline{a'} = \frac{1}{n}\sum_{1}^{n} a'_i = \frac{1}{n}\sum_{1}^{n}(a_i - \overline{a}) = \frac{1}{n}\sum_{1}^{n} a_i - \frac{1}{n}\sum_{1}^{n}\overline{a} = \overline{a} - \overline{a}$$

$$\overline{a'} = 0$$

Original Equation 3.5:

$$\overline{\overline{b}a'} = \frac{1}{n}\sum_{1}^{n} \overline{b}a' = \overline{b}\,\frac{1}{n}\sum_{1}^{n} a' = \overline{b}\,\overline{a'} = 0$$

Approach 2:

For a variable $a(t)$, the Reynolds averaging operation can be represented as:

$$\frac{1}{\Delta T}\int_{t}^{t+\Delta T} a(t)dt$$

For Original Equation 3.3:

$$\begin{aligned}
\overline{\overline{a}} &= \frac{1}{\Delta T}\int_t^{t+\Delta T}\left[\frac{1}{\Delta T}\int_t^{t+\Delta T} a(t)dt\right]dt \\
&= \frac{1}{\Delta T}\int_t^{t+\Delta T} \overline{a}dt \\
&= \overline{a}\frac{1}{\Delta T}\int_t^{t+\Delta T} dt \\
&= \overline{a}(1) \\
&= \overline{a}
\end{aligned}$$

For Original Equation 3.4, the Reynolds decomposition of a variable is defined as:

$$\begin{aligned}
a &= \overline{a} + a' \\
\text{Averaging :}& \\
\overline{a} &= \overline{\overline{a} + a'} \\
&= \overline{\overline{a}} + \overline{a'} \\
&= \overline{a} + \overline{a'} \\
\text{Rearranging :}& \\
\overline{a'} &= \overline{a} - \overline{a} \\
&= 0
\end{aligned}$$

For Original Equation 3.5:

$$\begin{aligned}
\overline{\overline{b}a'} &= \overline{a}\overline{a'} \text{ [Since } \overline{a} \text{ is a constant]} \\
&= \overline{a} \times 0 \\
&= 0
\end{aligned}$$

Table 3.1: (Original Table 3.1) Time series of temperature T (°C) and vertical velocity w ($\mathrm{m\,s^{-1}}$). Time t is in s.

t	1	2	3	4	5	6	7	8	9	10	11	12
T	20.0	20.2	19.7	19.8	20.1	20.3	20.3	20.2	20.2	19.7	20.0	20.0
w	−.52	−0.02	−.09	0.07	−.01	−.13	−.41	−.60	−.72	−.35	−.26	−.45

t	13	14	15	16	17	18	19	20	21	22	23	24
T	20.1	22.3	20.1	19.9	20.0	20.0	20.0	20.1	20.6	20.7	20.5	20.8
w	0.32	0.00	−.10	0.43	−.42	0.25	0.41	0.74	1.29	1.31	1.60	1.36

3.2 Calculate the variance of temperature (T) and vertical velocity (w) and the $T - w$ covariance using the time series data in Original Table 3.1.

Variance term is $\overline{a'^2} = \frac{1}{n}\sum_1^n a_i'^2$

Variance of temperature T:

$\overline{T} = \frac{485.6}{24} = 20.2°\mathrm{C}$

$\sum_1^{24} T_i'^2 = (T_1 - \overline{T})^2 + (T_2 - \overline{T})^2 + (T_3 - \overline{T})^2 + ... + (T_{24} - \overline{T})^2$

$= (20.0 - 20.2)^2 + (20.2 - 20.2)^2 + (19.7 - 20.2)^2 + ... + (20.8 - 20.2)^2 = 6.3$

$\overline{T'^2} = \frac{1}{24}\sum_1^{24} T_i'^2 = \frac{1}{24}(6.3) = 0.26\ \mathrm{K}^2$

Variance of vertical velocity w:

$\overline{w} = \frac{3.7}{24} = 0.15\ \mathrm{m\,s^{-1}}$

$\sum_1^{24} w_i'^2 = (w_1 - \overline{w})^2 + (w_2 - \overline{w})^2 + (w_3 - \overline{w})^2 + ... + (w_{24} - \overline{w})^2$

$= (-0.52 - 0.15)^2 + (-0.02 - 0.15)^2 + (-0.09 - 0.15)^2 + ... + (1.36 - 0.15)^2 = 10.2$

$\overline{w'^2} = \frac{1}{24}\sum_1^{24} w_i'^2 = \frac{1}{24}(10.2) = 0.43\ \mathrm{m^2\ s^{-2}}$

Covariance term is $\overline{a'b'} = \frac{1}{n}\sum_1^n a_i'b_i'$

$\overline{T'w'} = \frac{1}{24}\sum_1^{24} T_i'w_i'$

$= (T_1'w_1' + T_2'w_2' + ... + T_{24}'w_{24}')/24$

$= ((T_1 - \overline{T})(w_1 - \overline{w}) + (T_2 - \overline{T})(w_2 - \overline{w}) + ... + (T_{24} - \overline{T})(w_{24} - \overline{w}))/24$

$= ((-0.23)(-0.67) + (-0.03)(-0.17) + ... + (0.57)(1.21))/24$

$\overline{T'w'} = \frac{1}{24}(2.5) = 0.102\ \mathrm{K\,m\,s^{-1}}$

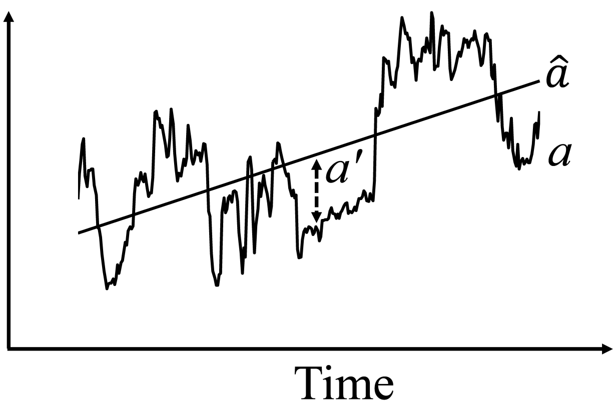

Figure 3.1: (Original Figure 3.7) Linear detrending of a time series

3.3* Some people define the fluctuating part of quantity a as,

$$a_i' = a_i - \hat{a}_i, \tag{3.1}$$

where subscript i denotes measurement at time step i, and $\hat{a}$ is the value from the linear regression of a against time (Figure 3.1). This procedure is called linear detrending. It can be shown that

$$\hat{a}_i = \overline{a} + b(t_i - \frac{1}{n}\sum_1^n t_i), \tag{3.2}$$

where n is the number of observations, t is time, and $b = b_1/b_2$, with b_1 and b_2 given by:

$$b_1 = \sum_1^n a_i t_i - \frac{1}{n}\sum_1^n a_i \sum_1^n t_i,$$

and

$$b_2 = \sum_1^n t_i t_i - \frac{1}{n}\sum_1^n t_i \sum_1^n t_i,$$

(Gash and Culf 1996). The detrended fluctuations are then used to compute the variance using Original Equation 3.9 and covariance using Original Equation 3.10. Show (i) that $\overline{a_i'} = 0$, and that (ii) the detrended variance and covariance are smaller in magnitude than their counterparts computed from the standard block averaging procedure. Verify these conclusions with the data given in Table 3.1.

(i) From Original Equations 3.80 and 3.81, we have

$$\overline{a_i'} = \overline{a} - \overline{\hat{a}_i} = \overline{a} - \left[\overline{a} + b\left(t_i - \overline{t}\right)\right] = -b\left(t_i - \overline{t}\right)$$

Because b is a constant:

$$\overline{a_i'} = -b\,\overline{\left(t_i - \overline{t}\right)} = -b\left(\overline{t} - \overline{t}\right) = 0$$

(ii) Detrended variance is given as:

$$\overline{a_i'^2} = \overline{\left(a_i - \hat{a}_i\right)^2} = \overline{a_i^2} - 2\overline{a_i\hat{a}_i} + \overline{\hat{a}_i^2}$$

Standard block variance is defined as:

$$\overline{a_i''^2} = \overline{\left(a_i - \overline{a}\right)^2} = \overline{a_i^2} - \overline{a}^2$$

Then the difference is:

$$
\begin{aligned}
&\overline{a_i'^2} - \overline{a_i''^2} = -2\overline{a_i \hat{a}_i} + \overline{\hat{a}_i^2} + \overline{a}^2 \\
&= -2\overline{a_i\left[\overline{a} + b\left(t_i - \overline{t}\right)\right]} + \overline{\left[\overline{a} + b\left(t_i - \overline{t}\right)\right]^2} + \overline{a}^2 \\
&= -2\overline{a}^2 - 2b\overline{a_i\left(t_i - \overline{t}\right)} + \overline{a}^2 + 2\overline{a}b\overline{\left(t_i - \overline{t}\right)} + b^2\overline{\left(t_i - \overline{t}\right)^2} + \overline{a}^2 \\
&= -2b\left(\overline{at} - \overline{a}\overline{t}\right) + b^2\overline{\left(t_i - \overline{t}\right)^2} \\
&= -2b \cdot b\left(\overline{t^2} - \overline{t}^2\right) + b^2\overline{\left(t_i - \overline{t}\right)^2} \\
&= -2b^2\overline{\left(t_i - \overline{t}\right)^2} < 0
\end{aligned}
$$

Therefore, the detrended variance is always smaller than the standard variance.

Detrended covariance:

$$
\overline{a_i' c_i'} = \overline{\left(a_i - \hat{a}_i\right)\left(c_i - \hat{c}_i\right)} = \overline{a_i c_i} - \overline{a_i \hat{c}_i} - \overline{\hat{a}_i c_i} + \overline{\hat{a}_i \hat{c}_i}
$$

Standard covariance:

$$
\overline{a_i'' c_i''} = \overline{\left(a_i - \overline{a}\right)\left(c_i - \overline{c}\right)} = \overline{a_i c_i} - \overline{a}\,\overline{c}
$$

$$
\begin{aligned}
&\overline{a_i' c_i'} - \overline{a_i'' c_i''} = -\overline{a_i \hat{c}_i} - \overline{\hat{a}_i c_i} + \overline{\hat{a}_i \hat{c}_i} + \overline{a}\,\overline{c} \\
&= -\overline{a_i\left[\overline{c} + d\left(t_i - \overline{t}\right)\right]} - \overline{\left[\overline{a} + b\left(t_i - \overline{t}\right)\right] c_i} + \overline{\left[\overline{a} + b\left(t_i - \overline{t}\right)\right]\left[\overline{c} + d\left(t_i - \overline{t}\right)\right]} + \overline{a}\,\overline{c} \\
&= -\overline{a}\,\overline{c} - d\overline{a_i\left(t_i - \overline{t}\right)} - \overline{ac} - b\overline{c_i\left(t_i - \overline{t}\right)} + \overline{ac} + d\overline{a}\overline{\left(t_i - \overline{t}\right)} + b\overline{c}\overline{\left(t_i - \overline{t}\right)} + bd\overline{\left(t_i - \overline{t}\right)^2} + \overline{a}\,\overline{c} \\
&= -d\left(\overline{at} - \overline{a}\overline{t}\right) - b\left(\overline{ct} - \overline{c}\overline{t}\right) + bd\overline{\left(t_i - \overline{t}\right)^2} \\
&= -d \cdot b\left(\overline{t^2} - \overline{t}^2\right) - b \cdot d\left(\overline{t^2} - \overline{t}^2\right) + bd\overline{\left(t_i - \overline{t}\right)^2} \\
&= -bd\overline{\left(t_i - \overline{t}\right)^2}
\end{aligned}
$$

In the above derivation, d is the linear slope of c versus t. Here whether the detrended covariance is smaller in magnitude than the standard covariance depends on the signs of b and d. If a and c are positively correlated, and if both show a linear increasing trend with time (i. e., $b > 0$ and $d > 0$) or a linear increasing trend with time ($b < 0$ and $d < 0$), the detrended covariance is lower than the standard variance. However this is not true if a and c are positively correlated and b and d have opposite signs.

(iii) The detrended and standard variances and covariance are computed from their respective fluctuations. Detrended w and T variances are 0.19 $\mathrm{m^2\,s^{-2}}$ and 0.23 $\mathrm{K^2}$, while standard w and T variances are 0.43 $\mathrm{m^2\,s^{-2}}$ and 0.26 $\mathrm{K^2}$. Detrended $w - T$ covariance is 0.02 $\mathrm{K\,m\,s^{-1}}$, while the standard covariance is 0.10 $\mathrm{K\,m\,s^{-1}}$.

3.4 The total kinetic energy per unit mass of air is given by

$$E_T = \frac{1}{2}(u^2 + v^2 + w^2).$$

Using the Reynolds averaging rules, show that the mean total kinetic energy, $\overline{E}_T$, is the sum of the mean flow kinetic energy, $\overline{E}$, and the turbulent kinetic energy, $\overline{e}$,

$$\overline{E}_T = \overline{E} + \overline{e},$$

where

$$\overline{E}_T = \frac{1}{2}\overline{(u^2 + v^2 + w^2)},$$
$$\overline{E} = \frac{1}{2}(\overline{u}^2 + \overline{v}^2 + \overline{w}^2),$$

and

$$\overline{e} = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}).$$

Here the Reynolds decomposition rules are used to obtain the above result:

$\overline{E_T} = \frac{1}{2}\overline{(u^2 + v^2 + w^2)}$

$u = \overline{u} + u'$, $v = \overline{v} + v'$, $w = \overline{w} + w'$

$\overline{E_T} = \frac{1}{2}\overline{[(\overline{u} + u')^2 + (\overline{v} + v')^2 + (\overline{w} + w')^2]}$

$= (\frac{1}{2})\overline{[(\overline{u}^2 + 2\overline{u}u' + u'^2) + (\overline{v}^2 + 2\overline{v}v' + v'^2) + (\overline{w}^2 + 2\overline{w}w' + w'^2)]}$

$= (\frac{1}{2})\overline{[(\overline{u}^2 + \overline{v}^2 + \overline{w}^2) + (u'^2 + v'^2 + w'^2) + (2\overline{u}u' + 2\overline{v}v' + 2\overline{w}w')]}$

Since $\overline{\overline{b}a'} = 0$, the third term $\overline{(2\overline{u}u' + 2\overline{v}v' + 2\overline{w}w')} = 0$, and we have

$\overline{E_T} = \frac{1}{2}\overline{(\overline{u}^2 + \overline{v}^2 + \overline{w}^2)} + \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)}$

Using $\overline{\overline{a}^2} = \overline{a}^2$ in the first term, we have $\overline{E_T} = \overline{E} + \overline{e}$

3.5 Show that the time averaging and the total derivative operations are not commutable, that is,

$$\overline{\frac{\mathrm{d}f}{\mathrm{d}t}} \neq \frac{\mathrm{d}\overline{f}}{\mathrm{d}t}. \tag{3.3}$$

Approach 1:

Let us consider a variable f which is a function of variables t, x and y, the latter two of which are functions of t. The total time derivative and its averaging are:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$\overline{\frac{df}{dt}} = \overline{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$
$$= \overline{\frac{\partial f}{\partial t}} + \overline{\frac{\partial f}{\partial x}\frac{dx}{dt}} + \overline{\frac{\partial f}{\partial y}\frac{dy}{dt}}$$

On the other hand, the total time derivative of the average is

$$\frac{d\overline{f}}{dt} = \frac{\partial \overline{f}}{\partial t} + \frac{\partial \overline{f}}{\partial x}\frac{dx}{dt} + \frac{\partial \overline{f}}{\partial y}\frac{dy}{dt}$$

which is not the same as the average of the total derivative.

Approach 2:

Let us consider the total derivative of mass density over time for the three dimensional case:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

Performing Reynolds averaging, we have:

$$\begin{aligned}
\overline{\frac{d\rho}{dt}} &= \overline{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}} \\
&= \overline{\frac{\partial \rho}{\partial t}} + \overline{u\frac{\partial \rho}{\partial x}} + \overline{v\frac{\partial \rho}{\partial y}} + \overline{w\frac{\partial \rho}{\partial z}} \\
&= \frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho u}}{\partial x} + \frac{\partial \overline{\rho v}}{\partial y} + \frac{\partial \overline{\rho w}}{\partial z} + \frac{\partial \overline{\rho' u'}}{\partial x} + \frac{\partial \overline{\rho' v'}}{\partial y} + \frac{\partial \overline{\rho' w'}}{\partial z}
\end{aligned}$$

On the other hand, the total time derivative of the average mass density is:

$$\frac{d\overline{\rho}}{dt} = \frac{\partial \overline{\rho}}{\partial t} + u\frac{\partial \overline{\rho}}{\partial x} + v\frac{\partial \overline{\rho}}{\partial y} + w\frac{\partial \overline{\rho}}{\partial z}$$

3.6 The horizontal velocity divergence $\partial\overline{u}/\partial x + \partial\overline{v}/\partial y$ is 2×10^{-6} s^{-1} in a mid-latitude anticyclone. Estimate the associated mean vertical velocity at the height of 20 m above the surface.

The mean vertical velocity is given by:

$$\begin{aligned}
\overline{w}(z) &= \int_0^z -(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y})dz' \\
&= \int_0^{20} -(2\times 10^{-6})dz' \\
&= -(2\times 10^{-6})\,[z]_0^{20} \\
&= -(2\times 10^{-6})\,[20-0] \\
&= -4\times 10^{-5}\ \text{m s}^{-1}
\end{aligned}$$

3.7 The momentum flux $\overline{u'w'}$ is -0.36 and 0 m^2 s^{-2} at the surface and at the top of a shear-dominated boundary layer (height = 1000 m), respectively. Find the vertical divergence of the turbulent momentum flux, $\partial\overline{u'w'}/\partial z$. Is the flux divergence much larger in magnitude than the molecular term, $\nu\nabla^2\overline{u}$? (Use your result for Problem 2.3 to answer this question.)

The vertical divergence of turbulent momentum flux, $\frac{\partial \overline{u'w'}}{\partial z}$ can be found using the given momentum fluxes at their respective heights:

$$\frac{\partial \overline{u'w'}}{\partial z} = \frac{0 + 0.36}{1000 - 0} = 3.6 \times 10^{-4}\ \mathrm{m\,s^{-2}} \tag{3.4}$$

The answers from Problem 2.3 state that the viscous force at 0.05 m is 7.7×10^{-3} $\mathrm{m\,s^{-2}}$ and at 10 m, 1.9×10^{-7} m s^{-2}. Based on this, the vertical divergence of turbulent momentum flux is comparable or smaller in size to the molecular term very near the ground, but quickly becomes significantly (orders of magnitude) larger as altitude increases.

3.8 Using the Reynolds rules and the weak incompressibility constraint (Original Equation 3.17), derive from Original Equation 2.20 the governing equation for the mean water vapor mixing ratio (Original Equation 3.28).

The conservation of water mass mixing ratio is given by:

$$\frac{\partial s_v}{\partial t} + u\frac{\partial s_v}{\partial x} + v\frac{\partial s_v}{\partial y} + w\frac{\partial s_v}{\partial z} = \frac{S_v}{\rho_d} + \kappa_v \nabla^2 s_v.$$

Applying Reynolds decomposition on the second term on the left, we have:

$$\begin{aligned} u\frac{\partial s_v}{\partial x} &= (\overline{u} + u')\frac{\partial(\overline{s_v} + s_v')}{\partial x} \\ &= \overline{u}\frac{\partial \overline{s_v}}{\partial x} + \overline{u}\frac{\partial s_v'}{\partial x} + u'\frac{\partial \overline{s_v}}{\partial x} + u'\frac{\partial s_v'}{\partial x} \end{aligned}$$

Using Reynolds averaging, we have:

$$\begin{aligned} \overline{u\frac{\partial s_v}{\partial x}} &= \overline{u}\frac{\partial \overline{s_v}}{\partial x} + \overline{u}\frac{\partial \overline{s_v'}}{\partial x} + \overline{u'}\frac{\partial \overline{s_v}}{\partial x} + \overline{u'\frac{\partial s_v'}{\partial x}} \\ &= \overline{u}\frac{\partial \overline{s_v}}{\partial x} + \overline{u'\frac{\partial s_v'}{\partial x}}, \end{aligned}$$

where both $\overline{u}\frac{\partial \overline{s_v'}}{\partial x}$ and $\overline{u'}\frac{\partial \overline{s_v}}{\partial x}$ equal zeros, because they are Reynolds means of fluctuations. Similarly, we can derive such equations on y and z directions:

$$\begin{aligned} \overline{v\frac{\partial s_v}{\partial y}} &= \overline{v}\frac{\partial \overline{s_v}}{\partial y} + \overline{v'\frac{\partial s_v'}{\partial y}} \\ \overline{w\frac{\partial s_v}{\partial z}} &= \overline{w}\frac{\partial \overline{s_v}}{\partial z} + \overline{w'\frac{\partial s_v'}{\partial z}} \end{aligned}$$

Using the product rule of derivative and weak incomprehensibility constraint, we have:

$$\begin{aligned} u'\frac{\partial s_v'}{\partial x} + v'\frac{\partial s_v'}{\partial y} + w'\frac{\partial s_v'}{\partial z} &= \frac{\partial u's_v'}{\partial x} + \frac{\partial v's_v'}{\partial y} + \frac{\partial w's_v'}{\partial z} - s_v'(\frac{\partial s_v'}{\partial x} + \frac{\partial s_v'}{\partial y} + \frac{\partial s_v'}{\partial z}) \\ &= \frac{\partial u's_v'}{\partial x} + \frac{\partial v's_v'}{\partial y} + \frac{\partial w's_v'}{\partial z} \\ \overline{u'\frac{\partial s_v'}{\partial x}} + \overline{v'\frac{\partial s_v'}{\partial y}} + \overline{w'\frac{\partial s_v'}{\partial z}} &= \frac{\partial \overline{u's_v'}}{\partial x} + \frac{\partial \overline{v's_v'}}{\partial y} + \frac{\partial \overline{w's_v'}}{\partial z} \end{aligned}$$

By taking average of the conversation equation and plugging in the above results, we have:

$$\frac{\partial \overline{s_v}}{\partial t} + \overline{u\frac{\partial s_v}{\partial x}} + \overline{v\frac{\partial s_v}{\partial y}} + \overline{w\frac{\partial s_v}{\partial z}} = \kappa_v \nabla^2 \overline{s_v}$$

$$\frac{\partial \overline{s_v}}{\partial t} + \overline{u}\frac{\partial \overline{s_v}}{\partial x} + \overline{v}\frac{\partial \overline{s_v}}{\partial y} + \overline{w}\frac{\partial \overline{s_v}}{\partial z} + \frac{\partial \overline{u's_v'}}{\partial x} + \frac{\partial \overline{v's_v'}}{\partial y} + \frac{\partial \overline{w's_v'}}{\partial z} = \kappa_v \nabla^2 \overline{s_v}$$

$$\frac{\partial \overline{s_v}}{\partial t} + \overline{u}\frac{\partial \overline{s_v}}{\partial x} + \overline{v}\frac{\partial \overline{s_v}}{\partial y} + \overline{w}\frac{\partial \overline{s_v}}{\partial z} = \kappa_v \nabla^2 \overline{s_v} - (\frac{\partial \overline{u's_v'}}{\partial x} + \frac{\partial \overline{v's_v'}}{\partial y} + \frac{\partial \overline{w's_v'}}{\partial z})$$

3.9 Derive from Original Equation 2.16 the governing equation from the mean mass density $\overline{\rho}_c$ (Original Equation 3.25). Do you need the weak incompressibility constraint for this derivation?

Original Equation 2.16 is the complete mass conservation for the mass density of CO_2 (ρ_c):

$$\frac{\partial \rho_c}{\partial t} + \frac{\partial u\rho_c}{\partial x} + \frac{\partial v\rho_c}{\partial y} + \frac{\partial w\rho_c}{\partial z} = S_c + \kappa_c \nabla^2 \rho_c \tag{3.5}$$

We note that:

$$u\rho_c = (\overline{u} + u')(\overline{\rho_c} + \rho_c')$$

Applying the Reynolds averaging rules, we have:

$$\overline{u\rho_c} = \overline{u}\,\overline{\rho_c} + \overline{u'\rho_c'}$$

So after Reynolds averaging, the second term on the left side of the conservation equation is:

$$\frac{\partial \overline{u\rho_c}}{\partial x} = \frac{\partial \overline{u}\,\overline{\rho_c}}{\partial x} + \overline{u'\rho_c'}$$

Similar expressions can be written for the third and fourth terms on the side of the conservation equation. Performing Reynolds averaging on the whole equation, we have

$$\frac{\partial \overline{\rho_c}}{\partial t} + \frac{\partial \overline{u\rho_c}}{\partial x} + \frac{\partial \overline{v\rho_c}}{\partial y} + \frac{\partial \overline{w\rho_c}}{\partial z} = \kappa_c \nabla^2 \overline{\rho_c} - (\frac{\partial \overline{u'\rho_c'}}{\partial x} + \frac{\partial \overline{v'\rho_c'}}{\partial y} + \frac{\partial \overline{w'\rho_c'}}{\partial z}) \tag{3.6}$$

This derivation does not require the weak incompressibility constraint.

3.10 Show that the eddy sensible heat flux, F_h, has the dimensions of $\mathrm{W\,m^{-2}}$ and the eddy water vapor flux, F_v, has the dimensions of $\mathrm{kg\,m^{-2}s^{-1}}$.

The eddy flux of sensible heat is:

$$F_h = \overline{\rho}_d c_p \overline{w'\theta'}$$

where $\overline{\rho_d}$ has units of (kg $\mathrm{m^{-3}}$), c_p has units of ($\mathrm{J\,kg^{-1}\,K^{-1}}$), and $\overline{w'\theta'}$ has units of ($\mathrm{m\,s^{-1}\,K}$). Therefore, the sensible heat flux has dimensions of $(\mathrm{kg\,m^{-3}})(\mathrm{J\,kg^{-1}\,K^{-1}})(\mathrm{m\,s^{-1}\,K}) = \mathrm{J\,m^{-2}\,s^{-1}}$ or $\mathrm{W\,m^{-2}}$.

The eddy water vapor flux is:

$$F_v = \overline{\rho}_d \overline{w' s'_v}$$

where $\overline{w' s'_v}$ has units of $(\mathrm{m\,s^{-1}})$. Therefore, the water vapor flux has dimensions of $\mathrm{kg\,m^{-2}\,s^{-1}}$.

3.11 In the situation shown in Original Figure 3.3, do you expect the $T - s_v$ covariance to be positive or negative? What about the $u - s_c$ covariance?

Based on the situation shown in Original Figure 3.3, both $T - s_v$ covariance and $u - s_c$ covariance are expected to be positive. In the former case, temperature and water vapor mixing ratio are both higher near the surface and decrease with altitude. Thus, an eddy will be both warmer and more humid than the surrounding air when it moves upward and, conversely, cooler and less humid when it moves downward.

In the case of horizontal velocity and CO_2 mixing ratio, both are smaller near the surface. Thus, an eddy moving upward will have a lower horizontal velocity and CO_2 mixing ratio than its surroundings, while the converse will be true for an eddy moving downward.

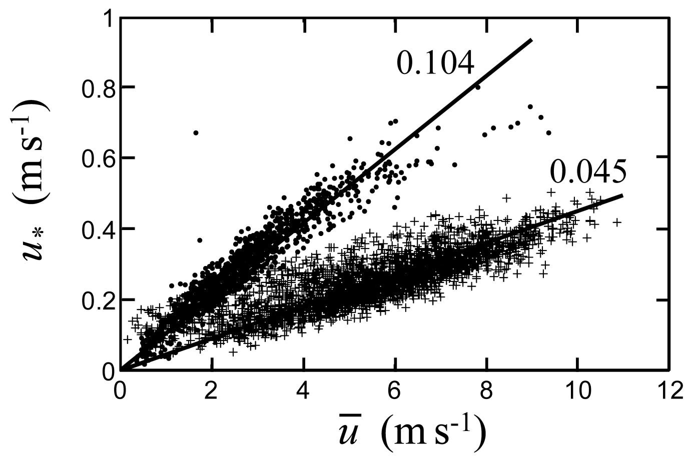

Figure 3.2: (Original Figure 3.8) Relationship between friction velocity (u_*) and horizontal velocity $\overline{u}$ over a lake surface and a wheat field. Solid lines represent regression fit to the data and numbers next to the lines indicate slopes of the regression.

3.12 Figure 3.2 shows the relationship between friction velocity (u_*) and horizontal velocity ($\overline{u}$) observed at a measurement height of 8.50 m above the surface of a shallow lake and over a wheat field at a measurement height of 2.55 m above its zero-plane displacement. Determine the surface momentum roughness and surface drag coefficient from the slope of the regression. Which dataset represents the lake experiment?

To solve this problem, the classic logarithmic wind profile is needed:

$$\overline{u} = \frac{u_*}{k} \ln \frac{z}{z_0} \tag{3.7}$$

Thus, u_* can be expressed as being proportional to $\overline{u}$,

$$u_* = \frac{k}{\ln \frac{z}{z_0}} \overline{u} \tag{3.8}$$

The part $\frac{k}{\ln \frac{z}{z_0}}$ is the slope shown in Figure 3.2.

Since the lake surface is smoother than the wheat surface, we expect its friction velocity to be lower. Using the smaller slope value,

$$\frac{k}{\ln \frac{z}{z_0}} = 0.045 \tag{3.9}$$

and plugging in the measurement height of 8.50 m for z, we obtain the z_0 for the lake to be 0.0012 m.

The z_0 value for wheat surface can be calculated by using $z = 2.55$ m and the larger slope value

$$\frac{k}{\ln \frac{2.55}{z_0}} = 0.104 \tag{3.10}$$

So we get $z_0 = 0.054$ m.

That the surface roughness of the lake is much lower than that of the wheat field confirms our initial guess that the slope of the regression of u_* against $\overline{u}$ should be lower for the lake surface than for the wheat surface.

Calculation of the drag coefficient C_D are based on these equations:

$$F_m = C_D \overline{u}^2, \qquad F_m = -\overline{u'w'}, \qquad (-\overline{u'w'})^{1/2} = u_*$$

Combining these three equations, we obtain the relationship for the drag coefficient as,

$$C_D^{1/2} = \frac{u_*}{\overline{u}} \tag{3.11}$$

So C_D for the lake and the wheat surface is 2.03×10^{-3} m and 1.08×10^{-2} m, respectively.

3.13 In some climate models, a grid cell can have multiple subgrid surface types. Each surface interacts independently with the overlaying atmosphere through forcing variables specified at the first model grid level. This level is usually at the so-called *blending height* where the atmosphere is well mixed horizontally (i. e., no sub-grid variations at this height). Assume that a grid cell consists of a smooth (momentum roughness $z_o = 0.001$ m) and a rough surface ($z_o = 0.50$ m), air stability is neutral, the blending height is 50 m, and wind speed at the blending height is 5.00 $\mathrm{m\,s^{-1}}$. Calculate the friction velocity for each of the two surfaces. What is the wind speed at the 2-m height above these surfaces?

As air stability is neutral, we can adopt Equation 3.7 to calculate the friction velocity $u_\star$.

For the smooth surface:

$$u_\star = 5.00 \times 0.4/\ln\frac{50}{0.001} = 0.18\ \mathrm{m\,s^{-1}}$$

For the rough surface:

$$u_\star = 5.00 \times 0.4/\ln\frac{50}{0.50} = 0.43\ \mathrm{m\,s^{-1}}$$

The wind speed at the 2-m height above the smooth surface is:

$$\overline{u} = 0.18/[0.4 \times \ln\frac{2}{0.001}] = 3.42\ \mathrm{m\,s^{-1}}$$

The wind speed at the same height above the rough surface is:

$$\overline{u} = 0.43/[0.4 \times \ln\frac{2}{0.001}] = 1.49\ \mathrm{m\,s^{-1}}$$

3.14 Two temperature sensors are mounted at heights of 2.0 and 4.0 m above the zero plane displacement of a grass surface whose momentum roughness is 0.02 m. The sensors have a precision of 0.05°C. Wind speed at the upper measurement height is 4.00 $\mathrm{m\,s^{-1}}$, and surface sensible heat flux is 35 $\mathrm{W\,m^{-2}}$. Assume that air stability is neutral. What is the temperature difference between the two measurement heights? Are these sensors good enough to resolve the difference? Repeat the calculation for a forest whose momentum roughness is 1.0 m. Can you resolve the temperature difference with the same sensors?

The vertical gradient of heat can be expressed with a flux-gradient method as:

$$F_h = -\bar{\rho}_d\ c_p\ K_h\ \frac{\overline{T}_2 - \overline{T}_1}{z_2 - z_1}$$

Here we assume STP state (273 K, 1 atm), and take the dry air mass density, $\bar{\rho}_d$, as 1.293 $\mathrm{kg\,m^{-3}}$, and specific heat of air at constant pressure, c_p, as 1004 $\mathrm{J\,kg^{-1}\,K^{-1}}$.

As neutral stability is assumed, K_h can be obtained as

$$K_h = kz_g u_*$$

and z_g, which is the geometric mean of the two measurement height, is defined as

$$z_g = [(z_2 - d)(z_1 - d)]^{1/2}$$

Meanwhile, the value for u_* can be obtained from

$$\bar{u} = \frac{u_*}{k} \ln \frac{z}{z_0}$$

where $k = 0.4$, z_0 is 0.02 m and $\bar{u}$ is 4.00 $\mathrm{m\,s^{-1}}$. The result for u_* is 0.30 $\mathrm{m\,s^{-1}}$. Then K_h is 0.33 $\mathrm{m^2\ s^{-1}}$.

Using the value given for F_h, we get a temperature difference of 0.163 °C.

In this case, a sensor with precision of 0.05 °C would lead to a potential relative error of 31%, which can not be neglected.

Repeating the above calculation for a forest with a surface roughness of 1.0 m, the temperature difference would be 0.041 °C. The precision of sensor is too low for this scenario.

3.15 Air temperature is 22.3 and 21.9°C and water vapor pressure is 18.1 and 17.4 hPa at heights of 1.0 m and 2.3 m above a soil surface, respectively. What is the Bowen ratio?

The Bowen ratio can be obtained from the flux-gradient relations,

$$\beta = \gamma \frac{\bar{T}_2 - \bar{T}_1}{\bar{e}_{v,2} - \bar{e}_{v,1}}$$

By substituting the parameters ($\gamma \approx 0.66$ $\mathrm{hPa\,K^{-1}}$ and $\bar{T}_2$, $\bar{T}_1$, $\bar{e}_{v,2}$, and $\bar{e}_{v,1}$), we get

$$\beta = 0.66 \times \frac{21.9 - 22.3}{17.4 - 18.1} = 0.377$$

So the Bowen ratio is 0.377.

3.16 The carbon dioxide flux of a lake system is on the order of 0.01 $\mathrm{mg\,m^{-2}s^{-1}}$. If a broadband carbon dioxide analyzer has a precision of 0.2 ppm, is it good enough for the flux-gradient measurement of the flux? Assume that the carbon dioxide concentration measurement takes place at heights of 1.0 m and 3.0 m above the water surface, the friction velocity is 0.15 $\mathrm{m\,s^{-1}}$ and air stability is neutral.

Approach 1:

We want to evaluate the smallest resolvable carbon dioxide flux F_c according to:

$$\frac{F_c}{\rho_d} = -K_c \frac{\overline{s}_{c,2} - \overline{s}_{c,1}}{z_2 - z_1}$$

where $(\overline{s}_{c,2} - \overline{s}_{c,1})$ is taken to be the analyzer's precision.

Note that $s_c = \frac{M_c}{M_d}\chi_c$ and under neutral stability, $K_c = K_m = kz_g u_* = k\sqrt{z_1\, z_2}\, u_*$, we have

$$\begin{aligned} F_c &= -\rho_d\, K_c\, \frac{\overline{s}_{c,2} - \overline{s}_{c,1}}{z_2 - z_1} \\ &= -\rho_d\, k\, \sqrt{z_1\, z_2}\, u_*\, \frac{\frac{44}{29}(\chi_{c,2} - \chi_{c,1})}{z_2 - z_1} \\ &= -1.25 \times 0.4 \times \sqrt{3} \times 0.15 \times \frac{\frac{44}{29} \times 2 \times 10^{-7}}{2} \\ &\approx 0.0197\ \mathrm{mg\, m^{-2}\, s^{-1}} > 0.01\ \mathrm{mg\, m^{-2}\, s^{-1}} \end{aligned}$$

Therefore, this carbon dioxide analyzer is not good enough for the flux-gradient measurement of the flux over the lake system.

Approach 2:

If the carbon dioxide molar mixing ratio at 3.0 m above the lake is 500 ppm or the mass mixing ratio of $(44/29 \times 500) = 758.6207\ \mathrm{mg\, kg^{-1}}$, consider the carbon dioxide concentration at 1.0 m. Original Equation 3.64 describes the vertical gradient over a finite distance:

$$F_c = -\overline{\rho}_d\, K_c\, \frac{\overline{s}_{c,2} - \overline{s}_{c,1}}{z_2 - z_1}$$

where

$$K_c = \frac{k\, z_g\, u_*}{\phi_h}$$

and

$$z_g = [(z_2 - d)(z_1 - d)]^{1/2}$$

In this problem $d = 0$ because there is no displacement plane, so $z_g = (3.0 \times 1.0)^{1/2} = \sqrt{3.0}$ m.

We note $\phi_h = 1$ because there is neutral stability, and $k = 0.4$ (von Karman constant), so

$$K_c = (0.4)(\sqrt{3.0}\ \mathrm{m})\ (0.15\ \mathrm{m\ s^{-1}}) = 0.104\ \mathrm{m^2\, s^{-1}}$$

Plugging the given order of magnitude of F_c, the calculated K_c, and an estimate of $\overline{\rho}_d$ of 1.2041 $\mathrm{kg\, m^{-3}}$ into Original Equation 3.64 results in:

$$0.01\, \mathrm{mg\, m^{-2}\, s^{-1}} = -1.2041\, \mathrm{kg\, m^{-3}}\ \times 0.104\, \mathrm{m^2\, s^{-1}} \frac{758.6207\, \mathrm{mg/kg} - \overline{s}_{c,1}}{3.0\, \mathrm{m} - 1.0\, \mathrm{m}}$$

Converting units and rearranging lead to:

$$0.01\, \mathrm{mg} = -47.4997\, \mathrm{mg} + 0.06261\, (\mathrm{kg})\, \overline{s}_{c,1}$$

and

$$\overline{s}_{c,1} = 758.8197\, \mathrm{mg\, kg^{-1}}$$

Converting to molar mixing ratio we obtain the CO_2 concentration expected for the lower height to be 500.13 ppm.

So a CO_2 analyzer with a precision of 0.2 ppm would not pick up the concentration difference between the two measurement heights (0.13 ppm) and thus is not good enough for the flux-gradient measurement of the flux.

3.17* Figure 3.3 shows the instant concentrations of three gases over a lake surface measured by a gas analyzer using the configuration shown in Original Figure 3.5. The step changes correspond to times when the analyzer switched from one air intake to another. The surface water vapor flux, measured by eddy covariance, is 0.082 $\mathrm{g\,m^{-2}s^{-1}}$ for the period shown in Figure 3.3. Using the modified Bowen ratio method, determine the surface flux of methane, in units of $\mathrm{\mu g\,m^{-2}s^{-1}}$, and the flux of carbon dioxide, in units of $\mathrm{mg\,m^{-2}s^{-1}}$ with the data shown in Figure 3.3.

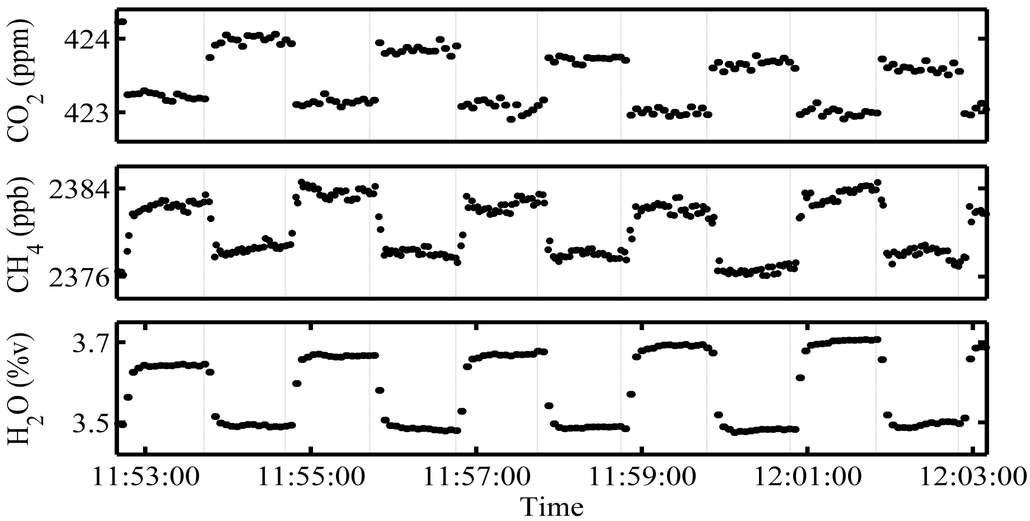

Figure 3.3: (Original Figure 3.9) Time series of carbon dioxide, methane and water vapor molar mixing ratio obtained with a modified Bowen ratio system. Data source: Xiao et al. (2014).

Assuming the equality of eddy diffusivity, the modified Bowen ratio method states that:

$$F_c = \frac{\overline{s}_{c,2} - \overline{s}_{c,1}}{\overline{s}_{v,2} - \overline{s}_{v,1}} F_v$$
$$F_{ma} = \frac{\overline{s}_{ma,2} - \overline{s}_{ma,1}}{\overline{s}_{v,2} - \overline{s}_{v,1}} F_v,$$

where subscripts v, c and ma denote water vapor, carbon dioxide and methane, respectively. The first time step in the switching sequence shown in Figure 3.3 is taken at the lower height z_1, the second time step is taken at the upper height z_2, and so on. We know this because the water vapor concentration should be greater at z_1 than at z_2. Evaluated from the Figure 3.3, we get the molar mixing ratios:

$$\overline{\chi}_{v,1} \approx 3.66 \times 10^{-2}, \qquad \overline{\chi}_{v,2} \approx 3.49 \times 10^{-2}$$

$$\overline{\chi}_{c,1} \approx 423.08 \times 10^{-6}, \qquad \overline{\chi}_{c,2} \approx 423.77 \times 10^{-6}$$

$$\overline{\chi}_{ma,1} \approx 2.383 \times 10^{-6}, \qquad \overline{\chi}_{ma,2} \approx 2.378 \times 10^{-6}$$

Converting molar mixing ratios to mass mixing ratios according to $s_c = \chi_c \frac{M_c}{M_d}$, we have:

$$\overline{s}_{v,1} \approx 2.27 \times 10^{-2}, \qquad \overline{s}_{v,2} \approx 2.17 \times 10^{-2}$$

$$\overline{s}_{c,1} \approx 641.91 \times 10^{-6}, \qquad \overline{s}_{c,2} \approx 642.96 \times 10^{-6}$$

$$\overline{s}_{ma,1} \approx 1.315 \times 10^{-6}, \qquad \overline{s}_{ma,2} \approx 1.312 \times 10^{-6}$$

Therefore:

$$F_c \approx \frac{642.96 \times 10^{-6} - 641.91 \times 10^{-6}}{2.17 \times 10^{-2} - 2.27 \times 10^{-2}} F_v \approx -0.086 \text{ mg m}^{-2}\text{s}^{-1}$$
$$F_c \approx \frac{1.312 \times 10^{-6} - 1.315 \times 10^{-6}}{2.17 \times 10^{-2} - 2.27 \times 10^{-2}} F_v \approx 0.25 \text{ µg m}^{-2}\text{s}^{-1}$$

3.18 Show that in neutral stability, the aerodynamic resistance to momentum transfer and to heat transfer are given by

$$r_{a,m} = \frac{1}{k^2\overline{u}}(\ln \frac{z-d}{z_o})^2, \tag{3.12}$$

and

$$r_{a,h} = \frac{1}{k^2\overline{u}} \ln \frac{z-d}{z_o} \ln \frac{z-d}{z_{o,h}}. \tag{3.13}$$

Assume that wind speed is measured at a reference height of 10.0 m above the zero plane displacement. Use the typical value of 0.14 for the roughness ratio $z_{o,h}/z_o$. Plot these resistances as a function of wind speed for a grass surface (momentum roughness $z_o = 0.10$ m) and for a forest ($z_o = 1.00$ m). Discuss how wind speed and surface roughness affect the aerodynamic resistances.

Under neutral stability, momentum and heat eddy diffusivity are parameterized as $K_m = K_h = k(z-d)u_*$ where $u_* = \sqrt{-\overline{u'w'}}$. Substitute into the flux-gradient equation:

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z}$$
$$-{u_*}^2 = -k(z-d)u_* \frac{\partial \overline{u}}{\partial z}$$
$$u_* = k(z-d)\frac{\partial \overline{u}}{\partial z}$$

Integrating from $z_0 + d$ to z,

$$\int_{z_0+d}^{z} \frac{u_*}{k}\frac{dz}{(z-d)} = \int_{z_0+d}^{z} d\overline{u}$$

we get the logarithmic mean profile:

$$\overline{u} = \frac{u_*}{k} \ln \frac{z-d}{z_0}$$

or equivalently:

$$u_* = \frac{k\overline{u}}{\ln \frac{z-d}{z_0}}.$$

The aerodynamic resistance to momentum is obtained as:

$$\begin{aligned}
r_{a,m} &= \int_{z_0+d}^{z} \frac{1}{K_m} dz' \\
&= \int_{z_0+d}^{z} \frac{1}{ku_*} \frac{1}{(z'-d)} dz' \\
&= \int_{z_0+d}^{z} \frac{1}{ku_*} d \ln (z'-d) \\
&= \frac{1}{ku_*} \ln \frac{z-d}{z_0} \\
&= \frac{1}{k^2\overline{u}} \left(\ln \frac{z-d}{z_0} \right)^2
\end{aligned}$$

Similarly, for $r_{a,h}$, employing $K_m = K_h$, we have:

$$\begin{aligned}
r_{a,h} &= \int_{z_{0,h}+d}^{z} \frac{1}{K_h} dz' \\
&= \int_{z_{0,h}+d}^{z} \frac{1}{ku_*} d \ln (z'-d) \\
&= \frac{1}{ku_*} \ln \frac{z-d}{z_{0,h}} \\
&= \frac{1}{k^2\overline{u}} \ln \frac{z-d}{z_0} \ln \frac{z-d}{z_{0,h}}.
\end{aligned}$$

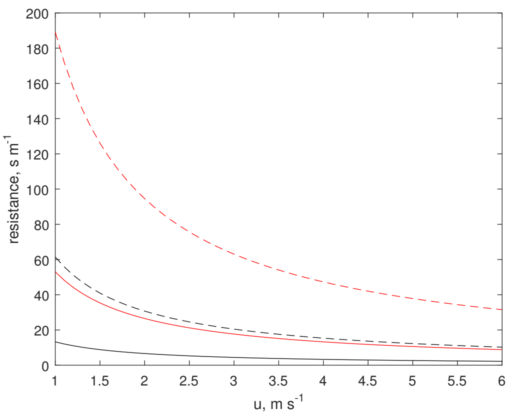

Figure 3.4: Aerodynamic resistance as a function of mean wind speed u: red solid line, $r_{a,m}$ for grass surface; red dashed line, $r_{a,h}$ for grass surface; black solid line, $r_{a,m}$ for forest; black dashed line, $r_{a,h}$ for forest.

It is clear from Figure 3.4 that aerodynamic resistance decreases with $\overline{u}$ in the form of $r \propto \frac{1}{\overline{u}}$. As the wind speed increases, shear-generated turbulence also increases and so does the efficiency of eddy diffusion, that is, aerodynamic resistance should decrease with increasing $\overline{u}$. As surface roughness z_0 increases, turbulent motion also strengthens and hence aerodynamic resistance should decrease with increasing z_0 or $z_{0,h}$.

3.19 In neutral stability, even though equality of eddy diffusivity holds for heat and for momentum (c. f., Original Equation 3.43) in the atmospheric surface layer, the aerodynamic resistance to heat transfer, $r_{a,h}$, is larger than that to momentum transfer, $r_{a,m}$. The difference, $r_e = r_{a,h} - r_{a,m}$, termed *excess resistance*, is explained by the fact that in the laminar layer in immediate contact with the surface, momentum transfer is much more efficient than heat transfer: the former accomplished by a from drag associated with pressure discontinuity, whereas the latter is carried out by molecular diffusion. Show that the excess resistance to heat transfer is given by

$$r_e = \frac{1}{ku_*} \ln \frac{z_o}{z_{o,h}}.$$

Using the information provided in Problem 3.18, compare r_e and $r_{a,h}$ for the grass surface and the forest surface over a wind speed range of 1 to 5 $\mathrm{m\,s^{-1}}$.

By definition and from the equations shown in Problem 3.18, we have:

$$\begin{aligned}
r_e &= r_{a,h} - r_{a,m} \\
&= \frac{1}{k^2\overline{u}} \ln \frac{z-d}{z_o} \ln \frac{z-d}{z_{o,h}} - \frac{1}{k^2\overline{u}} \big(\ln \frac{z-d}{z_o} \big)^2 \\
&= \frac{1}{k^2\overline{u}} \ln \frac{z-d}{z_o} \big(\ln \frac{z-d}{z_{o,h}} - \ln \frac{z-d}{z_o} \big)
\end{aligned}$$

Original Equation 3.47 allows us to substitute for $\overline{u}$:

$$\overline{u} = \frac{u_*}{k} \ln \frac{z}{z_o}$$

$$r_e = \frac{k}{k^2 u_* \ln \frac{z-d}{z_o}} \ln \frac{z-d}{z_o} \left(\ln \frac{\frac{z-d}{z_{o,h}}}{\frac{z-d}{z_o}} \right)$$

$$r_e = \frac{1}{ku_*} \big(\ln \frac{z_o}{z_{o,h}} \big)$$

In general, excess resistance decreases with increasing wind speed because the difference between resistance to momentum transfer and resistance to heat transfer decreases (Figure 3.5). Excess resistance is generally smaller than the aerodynamic resistance to heat diffusion. Excess resistance is is lower for the forest and for the grass, indicating that this resistance increases as the surface become smoother.

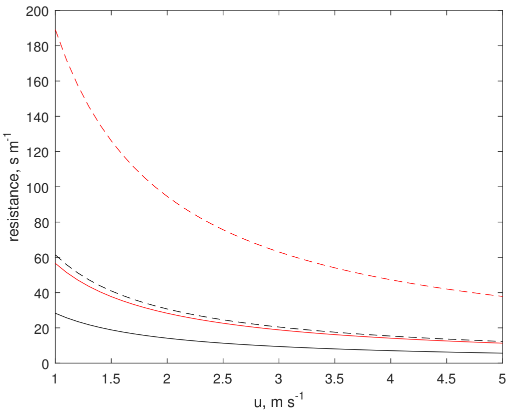

Figure 3.5: Resistance to heat diffusion as a function of mean wind speed u: red solid line, r_e for grass surface; red dashed line, $r_{a,h}$ for grass surface; black solid line, r_e for forest; black dashed line, $r_{a,h}$ for forest.

3.20 A typical value for C_H and C_E of lakes and oceans is 1×10^{-3}. The surface temperature of a lake is 18.0°C, and air at the height of 10.0 m above the lake has a temperature of 17.0°C, a relative humidity of 65% and a wind speed of 4.0 m s^{-1}. The lake is at the mean sea level. Calculate the surface sensible heat flux and the latent heat flux. The saturation vapor pressure is given by

$$e_v^* = 6.1365 \exp\left(\frac{17.502t}{240.97 + t}\right), \tag{3.14}$$

(Licor 2001), where e_v^* is in hPa and t is in °C.

As the pressure difference between lake surface (sea level pressure) and the height of 10 m is small, we can take potential temperature to be roughly equal to air temperature. Then the sensible heat flux is:

$$\begin{aligned} F_h &= \overline{\rho}_d\, c_p\, C_H\, \overline{u}(\overline{\theta}_0 - \overline{\theta}) \\ &\approx \overline{\rho}_d\, c_p\, C_H \overline{u}(\overline{T}_0 - \overline{T}) \\ &= 1.25 \text{ kg m}^{-3} \times 1004 \text{ J kg}^{-1} \text{ K}^{-1} \times 10^{-3} \times 4 \text{ m s}^{-1} \times 1 \text{ K} \\ &= 5.0 \text{ W m}^{-2}. \end{aligned}$$

According to the Clausius-Clapeyron Equation, vapor pressure $e_{v,0}$ at the lake surface, where $t = 18.0$°C, is about 20.71 hPa. At the height of 10 m, where $t = 17.0$°C, $e_{v,1} = 65\% \times 6.1365 \exp\left(\frac{17.502t}{240.97+t}\right) \approx 12.64$hPa. Converting molar mixing ratio to mass mixing ratio according to $s_v = \chi_v \frac{M_v}{M_d}$, one gets $s_{v,0} \approx 0.0127$ and $s_{v,1} \approx 0.0077$. Then the latent heat flux is:

$$\begin{aligned}
\lambda F_v &= \lambda\,\overline{\rho}_d\,C_E\,\overline{u}\,(\overline{s}_{v,0} - \overline{s}_{v,1}) \\
&= 2.5 \times 10^6 \text{ J kg}^{-1} \times 1.25 \text{ kg m}^{-3} \times 10^{-3} \times 4 \text{ m s}^{-1} \times (0.005) \\
&\approx 62.5 \text{ W}\,\text{m}^{-2}.
\end{aligned}$$

Chapter 4

Generation and Maintenance of Atmospheric Turbulence

4.1 Generate a synthetic time series dataset for the three instant velocity components. Calculate the mean total kinetic energy, the mean flow kinetic energy and the turbulent kinetic energy. Verify that your results satisfy Original Equation 4.2.

Approach 1:

Let $V = \{u, v, w\} = \{\sin t, 2\cos t, \sin 2t\}$ (in $\mathrm{m\,s^{-1}}$), where t falls in the range $0 \leq t \leq 100$. The total kinetic energy $\overline{E}_T$ is

$$\overline{E}_T = \frac{1}{200}\int_0^{100} \left(\sin^2 t + 4\cos^2 t + \sin^2 2t\right) dt = 1.497 \text{ m}^2 \text{ s}^{-2}$$

The mean velocities (in $\mathrm{m\,s^{-1}}$) are:

$$\overline{u} = \frac{1}{100}\int_0^{100} \sin t dt = 0.00137$$

$$\overline{v} = \frac{2}{100}\int_0^{100} \cos t dt = -0.0101$$

$$\overline{w} = \frac{1}{100}\int_0^{100} \sin 2t dt = 0.00256.$$

So the mean flow kinetic energy is

$$\overline{E} = \frac{1}{2}\left(\overline{u}^2 + \overline{v}^2 + \overline{w}^2\right) = \frac{1}{2}\left(0.00137^2 + 0.0101^2 + 0.00256^2\right) = 0.0000552 \text{ m}^2 \text{ s}^{-2}$$

Because

$$\overline{u'^2} = \frac{1}{100}\int_0^{100} \left(\sin t - 0.0014\right)^2 dt = 0.502 \text{ m}^2 \text{s}^{-2}$$

$$\overline{v'^2} = \frac{1}{100}\int_0^{100} \left(2\cos t + 0.010\right)^2 dt = 1.991 \text{ m}^2 \text{s}^{-2}$$

$$\overline{w'^2} = \frac{1}{100}\int_0^{100} \left(\sin 2t - 0.0025\right)^2 dt = 0.501 \text{ m}^2 \text{s}^{-2}$$

we have for the turbulent kinetic energy

$$\overline{e} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \frac{1}{2}(0.502 + 1.991 + 0.501) = 1.497 \text{ m}^2 \text{ s}^{-2}$$

The above result shows that $\overline{E}_T = \overline{E} + \overline{e}$.

Although this numerical example validates Original Equation 4.2, it is not realistic because in the boundary layer $\overline{E}$ is typically greater than $\overline{e}$.

Approach 2:

Let $u = \{3 \text{ m s}^{-1}, 4 \text{ m s}^{-1}, 5 \text{ m s}^{-1}\}$, $v = \{1 \text{ m s}^{-1}, 2 \text{ m s}^{-1}, 3 \text{ m s}^{-1}\}$, and $w = \{6 \text{ m s}^{-1}, 5 \text{ m s}^{-1}, 4 \text{ m s}^{-1}\}$.

The mean total kinetic energy can be calculated as,

$$\overline{E}_T = \frac{1}{2}\overline{(u^2 + v^2 + w^2)} = \frac{1}{2}\left[\frac{1}{3}(3^2 + 1^2 + 6^2 + 4^2 + 2^2 + 5^2 + 5^2 + 3^2 + 4^2)\right] = \frac{47}{2} \text{ m}^2 \text{ s}^{-2}$$

The mean flow kinetic energy is

$$\overline{E} = \frac{1}{2}(\overline{u}^2 + \overline{v}^2 + \overline{w}^2) = \frac{45}{2} \text{ m}^2 \text{ s}^{-2}$$

Subtracting the means from the instant variables, we get $u' = \{-1 \text{ m s}^{-1}, 0 \text{ m s}^{-1}, 1 \text{ m s}^{-1}\}$, $v' = \{1 \text{ m s}^{-1}, 0 \text{ m s}^{-1}, 1 \text{ m s}^{-1}\}$, and $w' = \{1 \text{ m s}^{-1}, 0 \text{ m s}^{-1}, -1 \text{ m s}^{-1}\}$. The turbulent kinetic energy is

$$\overline{e} = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) = 1 \text{ m}^2 \text{ s}^{-2}$$

Thus, we have:

$$\overline{E} + \overline{e} = \frac{45}{2} + 1 = \frac{47}{2} = \overline{E}_T$$

These results satisfy Original Equation 4.2.

4.2 The friction velocity is 0.32 m s^{-1}, and stability is neutral. (a) Calculate the shear destruction and viscous dissipation of MKE, and (b) the viscous dissipation of TKE, at the height of 10.0 m and 0.5 m in the surface layer. Assume that the TKE transport terms are negligible, the atmosphere is at steady state, and the mean velocity profile can be described by the logarithmic model with a momentum roughness of 0.1 m (Original Equation 2.50). How much larger is the viscous dissipation of TKE than the viscous dissipation of MKE?

When the stability is neutral:

$$-\overline{u'w'} = {u_\star}^2 = kzu_\star\frac{\partial \overline{u}}{\partial z}$$

The shear destruction of MKE can be expressed as

$$\overline{u'w'}\frac{\partial \overline{u}}{\partial z} = -\frac{{u_\star}^3}{kz}$$

So the shear destruction of MKE at heights of 10 m and 0.5 m is $-\frac{0.32^3}{0.4\times 10} = -0.0082\ \mathrm{m^2\,s^{-3}}$ and $-\frac{0.32^3}{0.4\times 0.5} = -0.16\ \mathrm{m^2\,s^{-3}}$, respectively.

The viscous dissipation of MKE is

$$\nu\overline{u}\nabla^2\overline{u} = \nu\overline{u}\frac{\partial^2 \overline{u}}{\partial z^2} = -\nu\big(\frac{u_\star}{kz}\big)^2\big(\ln\frac{z}{z_0}\big)$$

The viscous dissipation of MKE at heights of 10 m and 0.5 m are $1.48\times 10^{-5}\times\frac{0.32^2}{0.4\times 10^2}\times\ln\frac{10}{0.1} = 4.36\ \times 10^{-7}\ \mathrm{m^2\,s^{-3}}$ and $1.48\times 10^{-5}\times\frac{0.32^2}{0.4\times 0.5^2}\times\ln\frac{0.5}{0.1} = 6.10\ \times 10^{-5}\ \mathrm{m^2\,s^{-3}}$.

At these heights, the shear production of TKE is equal to $0.0082\ \mathrm{m^2\,s^{-3}}$ and $0.16\ \mathrm{m^2\,s^{-3}}$, respectively.

In the TKE budget equation (Original Equation 4.21), steady state means that the time rate of change term is equal to zero. Because the air is in neutral stability, the second term on the right side of this equation is equal to zero. We are told that the third and fourth therms are also negligible. With these approximations, the viscous dissipation of TKE is balanced by shear production

$$v\overline{u'\nabla^2 u'} = \overline{u'w'}\frac{\partial \overline{u}}{\partial z} = \frac{{u_\star}^3}{kz}$$

Therefore, the viscous dissipation of TKE at heights of 10 m and 0.5 m are 10^4 and 10^3 times larger than the viscous dissipation of MKE.

4.3 The normalized TKE dissipation in the surface layer is defined as

$$\phi_\epsilon = \frac{kz\epsilon}{u_*^3}.$$

Obtain an expression for ϕ_ϵ as a function of the Monin-Obukhov stability parameter ζ from the surface layer TKE budget equation (Original Equation 4.21), assuming that the transport terms are negligible and the atmosphere is in steady state. Plot this expression for the stability range $-1.0 < \zeta < 0.2$. Some field observations show that $\phi_\epsilon = 1.24$ in neutral stability. If this is true, is TKE transported into or out of the surface layer?

In steady state, and ignoring the the transport terms are negligible, the TKE budget equation (Original Equation 4.21) can be simplified as

$$-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} + \frac{g}{\overline{\theta}}\overline{w'\theta'} = \epsilon \tag{4.1}$$

According to Original Equations 4.33 and 3.45:

$$-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} = \frac{{u_\star}^3\phi_m}{kz}$$

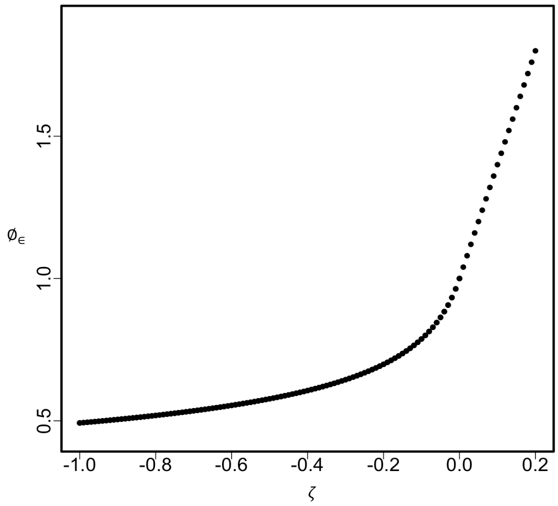

Figure 4.1: Normalized TKE dissipation rate ϕ_ϵ as a function of the Monin-Obukhov stability parameter ζ

$$\zeta = -\frac{kzg}{{u_\star}^3 \overline{\theta}} \overline{w'\theta'}$$

. Combining the above equations, we have:

$$\epsilon = \frac{{u_\star}^3 \phi_m}{kz} - \frac{{u_\star}^3}{kz} \zeta$$

This equation can be rearranged to

$$\phi_\epsilon = \frac{kz\epsilon}{{u_\star}^3} = \phi_m - \zeta$$

Or

$$\phi_\epsilon(\zeta) = \begin{cases} (1 - 16\zeta)^{-\frac{1}{4}} - \zeta & -1.0 < \zeta < 0 \\ 1 + 4\zeta & 0 \leq \zeta \leq 0.2 \end{cases} \tag{4.2}$$

Equation 4.2 indicates that the normalized TKE dissipation rate is a function of ζ only. A graphic representation of this equation is given in Figure 4.1.

Equation 4.2 indicates that in neutral stability $\phi_\epsilon = 1$. If the observed $\phi_\epsilon(\zeta)$ is greater than 1, it suggests that the two terms on the left side of Equation 4.1 underestimate the total source of TKE in the surface layer, implying that the TKE is transported into the surface layer.

4.4 The horizontal pressure gradient is 0.02 $\mathrm{hPa\,km^{-1}}$. The horizontal velocity is 10.0 $\mathrm{m\,s^{-1}}$. The angle between the horizontal velocity vector and the pressure gradient force vector is 60°. Calculate the MKE production.

The MKE production rate is:

$$
\begin{aligned}
-\frac{\overline{v}}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} &= \frac{1}{\overline{\rho}}|\boldsymbol{V}||-\nabla_H \overline{p}|\cos\alpha \\
&= \frac{1}{1.25\text{kg}\,\text{m}^{-3}} \times 10 \text{ m}\,\text{s}^{-1} \times 2 \times 10^{-3} \text{ Pa}\,\text{m}^{-1} \times \cos(60^\circ) \\
&= 0.8 \times 10^{-2} \text{ m}^2\,\text{s}^{-3}
\end{aligned}
$$

4.5 Calculate column total rates of shear and buoyancy production of TKE, in units of $\text{W}\,\text{m}^{-2}$, for the convective boundary layer depicted in Original Figure 4.5 (right panel). Assume that the depth of the boundary layer is 1000 m and the air density is 1.20 $\text{kg}\,\text{m}^{-3}$.

The known column-mean shear rate of TKE generation is 0.06×10^{-2} $\text{m}^2\,\text{s}^{-3}$. Thus, the total column rate is $0.06 \times 10^{-2} \times 1.20 \times 1000 = 0.72$ $\text{W}\,\text{m}^{-2}$.

The known column-mean buoyancy rate of TKE generation is 0.31×10^{-2} m^2 s^{-3}. Thus, the total column rate is $0.31 \times 10^{-2} \times 1.20 \times 1000 = 3.72$ $\text{W}\,\text{m}^{-2}$.

These energy terms are generally omitted from the surface and boundary layer energy balance analysis.

4.6 It has been frequently observed that a convective boundary layer collapses quickly in late afternoon after the surface stops producing sensible heat flux. The collapse time can be approximated by the time needed for viscous dissipation to consume all the TKE. Estimate the collapse time using the data shown in Original Figure 4.5 for the buoyancy-driven boundary layer. In your calculation, ignore shear and buoyancy production of TKE after the surface sensible heat flux has vanished and assume that viscous dissipation continues at the initial rate of 0.37×10^{-2} $\text{m}^2\,\text{s}^{-3}$.

The collapse time can be calculated by dividing TKE by viscous dissipation rate:

$$
\Delta t = \frac{1.8\,\text{m}^2\text{s}^{-2}}{0.37 \times 10^{-2}\,\text{m}^2\text{s}^{-3}} = 486 \text{ s} \simeq 8 \text{ min}
$$

4.7 Improve the collapse time calculation in Problem 4.6 by deploying a time-dependent parameterization for the dissipation term, as

$$
\epsilon = \frac{\overline{e}^{3/2}}{\Lambda} \tag{4.3}
$$

where Λ is a length scale and is related to the eddy mixing length (Equation **??**; Figure **??**) as

$$
\Lambda = Bl
$$

with the coefficient $B = 5.0$.

Because shear and buoyancy cease to produce turbulence, we have:

$$\frac{\partial \overline{e}}{\partial t} = -\epsilon = -\frac{\overline{e}^{3/2}}{Bl}$$

or

$$\overline{e}^{-3/2}\frac{\partial \overline{e}}{\partial t} = -\frac{1}{Bl}$$

On the approximation that l is a constant, we can integrate the above equation to obtain:

$$-2([\overline{e}(t)]^{-1/2} - [\overline{e}(0)]^{-1/2}) = -\frac{t}{Bl}$$

or

$$\overline{e}(t) = \left([\overline{e}(0)]^{-1/2} + t/(2Bl)]\right)^{-2} \tag{4.4}$$

According to Original Equation 3.51, for a boundary layer height of 1000 m, the mean mixing length is about 33 m. Plugging in the initial value $\overline{e}(0) = 1.8\ \mathrm{m^2\,s^{-2}}$ and $B = 5$, Equation 4.4 shows that the TKE will drop to 1% of the initial value at about $t = 2000$ s or 33 min. This collapse time is longer than given in Problem 4.6 but is still very fast.

4.8 Let us define a dimensionless TKE as

$$\phi_e = \frac{\overline{e}}{u_*^2}.$$

Obtain an expression for ϕ_e as a function of the Monin-Obukhov stability parameter ζ for the surface layer from the parameterization equation (Original Equation 4.39) and the surface TKE budget equation (Original Equation 4.21). Assume in your derivation that the air is at steady state and that the transport terms are negligible. Use the expression to estimate $\overline{e}$ for the following conditions: (1) $u_* = 0.28\ \mathrm{m\,s^{-1}}$, $\zeta = -0.5$; (2) $u_* = 0.28\ \mathrm{m\,s^{-1}}$, $\zeta = 0$; (3) $u_* = 0.15\ \mathrm{m\,s^{-1}}$, $\zeta = 0$; (4) $u_* = 0.15\ \mathrm{m\,s^{-1}}$, $\zeta = 0.2$. Comment on how friction velocity and air stability affect TKE in the surface layer.

Approach 1:

Let us assume that the mixing length in the surface layer takes the form for neutral stability $l = kz$. Combining Original Equation 4.21 and Equation 4.3 and ignoring transport terms, we have

$$\frac{\overline{e}^{3/2}}{Bkz} = -\overline{u'w'}\frac{\partial \overline{u}}{\partial z} + \frac{g}{\overline{\theta}}\overline{w'\theta'} \tag{4.5}$$

The left side of the above equation can be written as:

$$\frac{u_*^3\phi_e^{3/2}}{Bkz}$$

Plugging this expression in Equation 4.5 and rearranging using a strategy similar to that used for Problem 4.3, we obtain:

$$\phi_e(\zeta) = [B(\phi_m(\zeta) - \zeta)]^{2/3}$$

The above equation can be used to estimate the TKE if u_* and ζ are known. The four numerical examples:

$$u_* = 0.28 \text{ m s}^{-1}, \zeta = -0.5, \phi_e = 3.07, \overline{e} = 0.24 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.28 \text{ m s}^{-1}, \zeta = 0, \phi_e = 2.92, \overline{e} = 0.23 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.15 \text{ m s}^{-1}, \zeta = 0, \phi_e = 2.92, \overline{e} = 0.066 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.15 \text{ m s}^{-1}, \zeta = 0.2, \phi_e = 4.33, \overline{e} = 0.097 \text{ m}^2 \text{ s}^{-2}$$

Under neutral stability, the TKE increases with increasing u_*. The effect of stability is weak. That the TKE is greater under stable conditions than under neutral conditions at the same u_* does not seem to make sense.

Approach 2:

A more accurate expression can be obtained by allowing the length scale to vary with stability. Original Equation 3.50 suggests that:

$$l = \frac{kz}{\phi_m}$$

The final result is

$$\phi_e(\zeta) = [B(1 - \zeta/\phi_m(\zeta))]^{2/3}$$

The four numerical examples:

$$u_* = 0.28 \text{ m s}^{-1}, \zeta = -0.5, \phi_e = 4.43, \overline{e} = 0.35 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.28 \text{ m s}^{-1}, \zeta = 0, \phi_e = 2.92, \overline{e} = 0.23 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.15 \text{ m s}^{-1}, \zeta = 0, \phi_e = 2.92, \overline{e} = 0.066 \text{ m}^2 \text{ s}^{-2}$$

$$u_* = 0.15 \text{ m s}^{-1}, \zeta = 0.2, \phi_e = 2.73, \overline{e} = 0.061 \text{ m}^2 \text{ s}^{-2}$$

These results are more reasonable than those obtained with Approach 1. The stability effect is stronger. At the same friction velocity, the TKE is greater in neutral conditions than in stable conditions.

4.9 The vertical TKE flux, $\overline{ew'}$, is 0.825 and 0.044 $m^3\,s^{-3}$ at the heights of 300 and 600 m above the ground, respectively, in a shear-driven boundary layer. Calculate the vertical turbulent TKE transport. Does the transport term contribute to a gain or a loss of local TKE in the 300 to 600 m air layer?

The total TKE change in this layer can be calculated by integrating the vertical transport term from 300 to 600 m:

$$\begin{aligned}\int_{300}^{600} \frac{\partial \overline{e}}{\partial t} dz &= \int_{300}^{600} -\frac{\partial \overline{ew'}}{\partial z} dz \\ &= (-\overline{ew'})_{|z=600} - (-\overline{ew'})_{|z=300} \\ &= (-0.044)\ \mathrm{m^3s^{-3}} - (-0.825)\ \mathrm{m^3s^{-3}} = 0.781\ \mathrm{m^3s^{-3}} > 0\end{aligned}$$

Therefore, the transport term contributes to a gain in this layer.

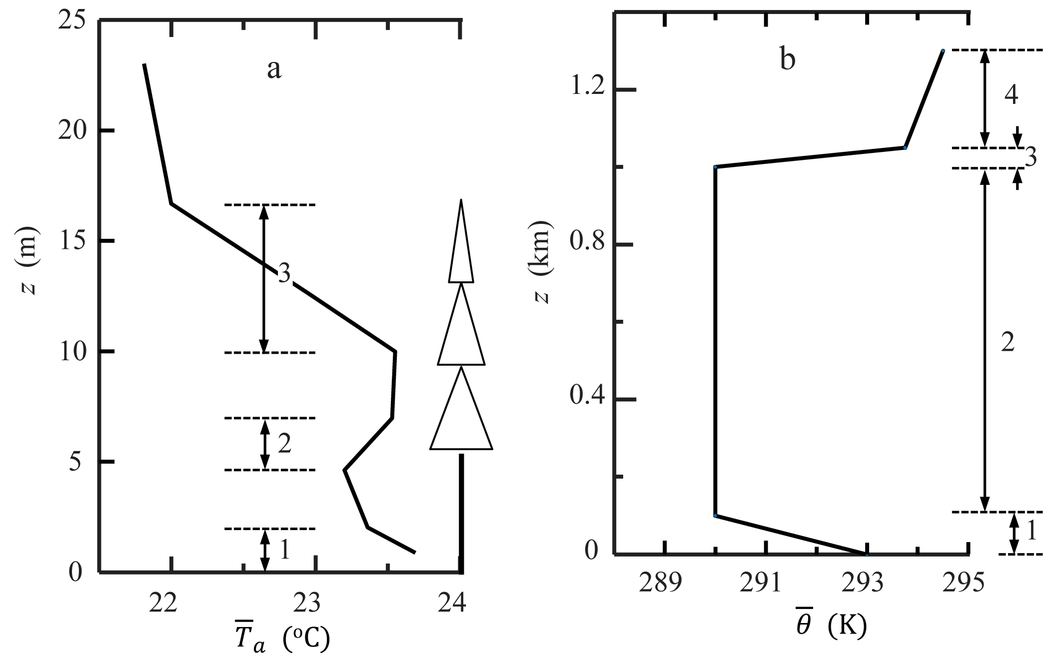

Figure 4.2: (Original Figure 4.10) Profile of air temperature in a forest at midday (a) and profile of potential temperature in a daytime convective boundary layer (b).

4.10 Determine if each of the air layers indicated in Figure 4.2 is unstable, neutral or stable.

In Figure 4.2a, we can judge the stability of each layer by comparing the air temperature change with height with the dry adiabatic lapse rate ($\sim 9.8\ \mathrm{K\,km^{-1}}$).

Layers	Actual lapse rate/K km^{-1}	Stability
(1)	-294	Unstable
(2)	137	Stable
(3)	-232	Unstable

In Figure 4.2b, the stability can be determined by the following method:

$$\frac{\partial\overline{\theta}}{\partial z} \begin{array}{ll} < 0 & \text{unstable} \\ = 0 & \text{neutral} \\ > 0 & \text{stable} \end{array}$$

Layer (1) is unstable, layer (2) is neutral, and layers (3) and (4) are stable.

4.11 Show that the flux Richardson number and the gradient Richardson number are dimensionless and that the Obukhov length has the dimension of length.

The flux Richardson number is given by:

$$R_f = \frac{\frac{g}{\theta}\overline{w'\theta'}}{\overline{u'w'}\frac{\partial\overline{u}}{\partial z} + \overline{v'w'}\frac{\partial\overline{v}}{\partial z}}$$

Replacing the variables with their dimensions:

$$\begin{aligned} &= \frac{\frac{\mathrm{L\ T^{-2}}}{\mathrm{K}}\mathrm{L\ T^{-1}\ K}}{\mathrm{L\ T^{-1}\ L\ T^{-1}}\frac{\mathrm{L\ T^{-1}}}{\mathrm{L}} + \mathrm{L\ T^{-1}\ L\ T^{-1}}\frac{\mathrm{L\ T^{-1}}}{\mathrm{L}}} \\ &= \frac{\mathrm{L^2\ T^{-3}}}{\mathrm{L^2\ T^{-3}}} \\ &= \text{dimensionless} \end{aligned}$$

The gradient Richardson number is given by:

$$R_i = \frac{\frac{g}{\theta}\frac{\partial\overline{\theta}}{\partial z}}{(\frac{\partial\overline{u}}{\partial z})^2 + (\frac{\partial\overline{v}}{\partial z})^2}$$

Replacing the variables with their dimensions:

$$\begin{aligned} &= \frac{\frac{\mathrm{L\ T^{-2}}}{\mathrm{K}}\frac{\mathrm{K}}{\mathrm{L}}}{(\frac{\mathrm{L\ T^{-1}}}{\mathrm{L}})^2 + (\frac{\mathrm{L\ T^{-1}}}{\mathrm{L}})^2} \\ &= \frac{\mathrm{T^{-2}}}{\mathrm{T^{-2}}} \\ &= \text{dimensionless} \end{aligned}$$

The Obukhov length is given by:

$$L = -\frac{u_*^3}{k(\frac{g}{\theta})\overline{w'\theta'}}$$

Replacing the variables with their dimensions:

$$= \frac{\mathrm{L}^3\,\mathrm{T}^{-3}}{(\frac{\mathrm{L}\,\mathrm{T}^{-2}}{\mathrm{K}})\mathrm{L}\,\mathrm{T}^{-1}\,\mathrm{K}}$$
$$= \mathrm{L}$$

Table 4.1: (Original Table 4.1) Observed friction velocity u_* ($\mathrm{m\,s^{-1}}$) and sensible heat flux F_h ($\mathrm{W\,m^{-2}}$) at a measurement height of 3.5 m above the surface of a shallow lake and at 15.2 m above the displacement height of a forest.

Time	Lake					Forest				
	u_*	F_h	ζ	K_m	K_h	u_*	F_h	ζ	K_m	K_h
00:10	0.09	0.1				0.14	21.6			
12:40	0.24	33.4				0.32	436.8			

4.12 Calculate the Monin-Obukhov stability parameter ζ and the eddy diffusivity for momentum (K_m) and for heat (K_h) using the data shown in Table 4.1.

Filling in the table requires a few key relationships. First, the eddy diffusivities for momentum, K_m, and heat K_h are

$$K_m = \frac{kzu_*}{\phi_m} \tag{4.6}$$

and

$$K_h = \frac{kzu_*}{\phi_h} \tag{4.7}$$

and the Monin-Obukhov stability parameter is given as

$$\zeta = -\frac{zk\frac{g}{\theta}\overline{w'\theta'}}{u_*^3} \tag{4.8}$$

Next, we note that $\overline{w'\theta'} = F_h/(\rho c_p)$, where $\rho c_p \simeq 1200\ \mathrm{J\,K^{-1}\,m^{-3}}$.

To calculate ζ, we also need $\overline{\theta}$, which is not given. But we can assign an approximate value of 290 K because ζ is not sensitive to $\overline{\theta}$.

Using these relationships, the remaining values can be found:

Site	time	ζ	ϕ_m	ϕ_h	K_m $\mathrm{m\,s^{-1}}$	K_h $\mathrm{m\,s^{-1}}$
Lake	00:10	-0.0054	0.98	0.96	0.13	0.13
Lake	12:40	-0.095	0.79	0.63	0.43	0.53
Forest	00:10	-1.35	0.46	0.21	1.85	4.13
Forest	12:40	-2.28	0.54	0.16	3.60	12.16

4.13* Integration of Original Equations 4.33 and 4.34 with respect to z yields

$$\frac{\overline{u}(z)}{u_*} = \frac{1}{k}[\ln\frac{z}{z_o} - \Psi_m(\zeta)], \tag{4.9}$$

$$\frac{\overline{\theta}(z)-\theta_o}{\theta_*} = \frac{1}{k}[\ln\frac{z}{z_{o,h}} - \Psi_h(\zeta)], \tag{4.10}$$

where

$$\Psi_m = \int_{z_o/L}^{\zeta} [1-\phi_m(\xi)]\frac{\mathrm{d}\xi}{\xi} \simeq \int_0^{\zeta} [1-\phi_m(\xi)]\frac{\mathrm{d}\xi}{\xi},$$

and:

$$\Psi_h = \int_{z_{o,h}/L}^{\zeta} [1-\phi_h(\xi)]\frac{\mathrm{d}\xi}{\xi} \simeq \int_0^{\zeta} [1-\phi_h(\xi)]\frac{\mathrm{d}\xi}{\xi}$$

are called integral similarity or stability functions (Paulson 1970). The mean wind and potential temperature profiles now deviate from the logarithmic relations (Original Equations 3.47 and 3.48) due to stability effects. Show that

$$\begin{aligned} \Psi_m = \Psi_h &= -5\,\zeta, \quad \zeta \geq 0 \\ \Psi_m = \ln[(\frac{1+x^2}{2})(\frac{1+x}{2})^2] - 2\arctan x + \frac{\pi}{2}&, \quad \zeta < 0 \\ \Psi_h = 2\ln\left(\frac{1+x^2}{2}\right)&, \quad \zeta < 0 \end{aligned} \tag{4.11}$$

where $x = (1-16\,\zeta)^{1/4}$.

The derivation can be found in: Paulson CA (1970) The mathematical representation of wind speed and temperature in the unstable atmospheric surface layer. J Appl Meteorol 9: 857-861.

4.14 Show from Original Equations 3.72, 3.74, and 3.79, and Equations 4.9 to 4.11 that the aerodynamic resistances and the transfer coefficients in stratified air can be expressed as,

$$r_{a,m} = \frac{1}{k^2\overline{u}}\,[\ln\frac{z}{z_o} - \Psi_m(\zeta)]^2, \tag{4.12}$$

$$r_{a,h} = \frac{1}{k^2\overline{u}}\,[\ln\frac{z}{z_o} - \Psi_m(\zeta)][\ln\frac{z}{z_{o,h}} - \Psi_h(\zeta)], \tag{4.13}$$

$$C_D = k^2[\ln\frac{z}{z_o} - \Psi_m(\zeta)]^{-2}, \tag{4.14}$$

$$C_H = k^2[\ln\frac{z}{z_o} - \Psi_m(\zeta)]^{-1}[\ln\frac{z}{z_{o,h}} - \Psi_h(\zeta)]^{-1}. \tag{4.15}$$

Calculate the resistances and the transfer coefficients for a grass (momentum roughness z_o = 0.02 m) and a bare soil (z_o = 0.002 m) at $\zeta = -0.5$, 0 and 0.2. Assume in your calculation that wind speed is 3.0 $\mathrm{m\,s^{-1}}$, the roughness ratio $z_{o,h}/z_o$ is 0.14, and reference height z is 5.0 m. Which surface is more sensitive to stability effects?

According to Original Equation 3.72, we have:

$$r_{a,m} = \frac{\overline{u}}{u_*^2}$$

Combining this equation with Equation 4.9, we obtain

$$\begin{aligned} r_{a,m} &= \frac{1}{ku_*}[\ln\frac{z}{z_o} - \Psi_m(\zeta)] \\ &= \frac{1}{k^2\overline{u}}[\ln\frac{z}{z_o} - \Psi_m(\zeta)]^2 \end{aligned}$$

Rearranging Original Equation 3.74 and making use of Original Equation 3.49 gives:

$$r_{a,h} = \frac{\overline{\theta}_o - \overline{\theta}}{\overline{w'\theta'}} = -\frac{\overline{\theta}_o - \overline{\theta}}{\theta_* u_*}$$

Combining this equation with Equations 4.9 and 4.10, we obtain

$$\begin{aligned} r_{a,h} &= \frac{1}{ku_*}[\ln\frac{z}{z_{o,h}} - \Psi_h(\zeta)] \\ &= \frac{1}{k^2\overline{u}}[\ln\frac{z}{z_o} - \Psi_m(\zeta)][\ln\frac{z}{z_{o,h}} - \Psi_h(\zeta)] \end{aligned}$$

Substituting Equations 4.12 and 4.13 in Original Equation 3.79, we obtain Equations 4.14 and 4.15.

The resistances and transfer coefficients for a grass and a bare soil can then be calculated as follows:
At ζ = -0.5:

$$\zeta < 0 \tag{4.16}$$

$$\Psi_m = \ln[(\frac{1+x^2}{2})(\frac{1+x}{2})^2] - 2\mathrm{arctan}x + \frac{\pi}{2} \tag{4.17}$$

$$\Psi_h = 2\ln\left(\frac{1+x^2}{2}\right) \tag{4.18}$$

where $x = (1-16\zeta)^{\frac{1}{4}}$. Aerodynamic resistances $r_{a,m}$ and $r_{a,h}$ for grass are 36.8 s m^{-1} and 5.08 s m^{-1}, respectively, and for soil, 88.2 s m^{-1} and 7.86 s m^{-1}, respectively. Transfer coefficients C_D and C_H are 0.00905 and 0.0221, respectively, for grass and 0.00378 and 0.0143, respectively, for soil.
At ζ = 0 and 0.2:

$$\zeta \geq 0 \tag{4.19}$$

$$\Psi_m = \Psi_h = -5\zeta \tag{4.20}$$

For $\zeta = 0$: Aerodynamic resistances $r_{a,m}$ and $r_{a,h}$ are found to be 63.5 s m^{-1} and 22.6 s m^{-1}, respectively, for grass and 128 s m^{-1} and 32.0 s m^{-1}, respectively, for soil. Transfer coefficients C_D and C_H are 4.88 and 0.0147, respectively, for grass and 9.79 and 0.0104, respectively, for soil.

Table 4.2: (Original Table 4.2) Horizontal velocities ($\overline{u}$ and $\overline{v}$) and potential temperature ($\overline{\theta}$) in an atmospheric boundary layer. Observation height is denoted by z.

z (m)	$\overline{u}$ ($\mathrm{m\,s^{-1}}$)	$\overline{v}$ ($\mathrm{m\,s^{-1}}$)	$\overline{\theta}$ (K)
0	0	0	287.0
100	6.9	0.8	285.2
200	7.3	1.0	284.6

Finally, for $\zeta = 0.2$: Aerodynamic resistances $\mathrm{r}_{a,m}$ and $\mathrm{r}_{a,h}$ are 42.6 s m^{-1} and 0.477 s m^{-1}, respectively, for grass and 97.0 s m^{-1} and 0.316 s m^{-1}, respectively, for soil. Transfer coefficients C_D and C_H are 0.00783 and 0.0366, respectively, for grass and 0.00344 and 0.0243, respectively, for soil.

4.15 Using the data in Table 4.2, compute the gradient Richardson number for the air layer between the heights of 0 and 100 m and between the heights of 100 and 200 m.

Between 0 and 100 m:

$$\begin{aligned} R_i &= \frac{(\frac{9.8}{286.1})(\frac{287-285.2}{0-100})}{(\frac{0-6.9}{0-100})^2 + (\frac{0-0.8}{0-100})^2} \\ &= \frac{-0.00062\ \mathrm{s}^2}{0.00476\ \mathrm{s}^2 + 0.000064\ \mathrm{s}^2} \\ &= -0.13 \end{aligned}$$

Between 100 and 200 m:

$$\begin{aligned} R_i &= \frac{(\frac{9.8}{284.9})(\frac{285.2-284.6}{100-200})}{(\frac{6.9-7.3}{100-200})^2 + (\frac{0.8-1}{100-200})^2} \\ &= \frac{-0.00021\ \mathrm{s}^2}{0.000016\ \mathrm{s}^2 + 0.000004\ \mathrm{s}^2} \\ &= -10.5 \end{aligned}$$

4.16 Show that the gradient Richardson number in the surface layer is related to the Monin-Obukhov stability parameter as,

$$R_i = \zeta \phi_h / \phi_m^2. \tag{4.21}$$

Plot this relationship over the stability range $-2 < \zeta < 0.5$.

To begin, a reminder that a potential temperature scale, θ_* is defined as:

$$\theta_* = -\frac{\overline{w'\theta'}}{u_*} \tag{4.22}$$

which can be rearranged to

$$\overline{w'\theta'} = -\theta_* u_* \tag{4.23}$$

Next, the Monin-Obukhov stability parameter is given as

$$\zeta = -\frac{zk\frac{g}{\theta}\overline{w'\theta'}}{u_*^3} \tag{4.24}$$

The covariance term can be replaced by the relationship shown previously:

$$\zeta = -\frac{zk\frac{g}{\theta}(-\theta_* u_*)}{u_*^3} \tag{4.25}$$

Original Equations 4.33 and 4.34 can be rearranged in terms of u_* and θ_*, respectively:

$$\phi_m = \frac{kz}{u_*}\frac{\partial \overline{u}}{\partial z} \tag{4.26}$$

$$u_* = \frac{kz}{\phi_m}\frac{\partial \overline{u}}{\partial z} \tag{4.27}$$

and

$$\phi_h = \frac{kz}{\theta_*}\frac{\partial \overline{\theta}}{\partial z} \tag{4.28}$$

$$\theta_* = \frac{kz}{\phi_h}\frac{\partial \overline{\theta}}{\partial z} \tag{4.29}$$

Substituting these into the Monin-Obukhov stability parameter equation derived above gives:

$$\zeta = \frac{\phi_m^2}{\phi_h}\frac{\frac{g}{\theta}\frac{\partial \overline{\theta}}{\partial z}}{(\frac{\partial \overline{u}}{\partial z})^2} \tag{4.30}$$

Original Equation 4.28, which gives the gradient Richardson number:

$$R_i = \frac{\frac{g}{\theta}\frac{\partial \overline{\theta}}{\partial z}}{(\frac{\partial \overline{u}}{\partial z})^2} \tag{4.31}$$

can be substituted in:

$$\zeta = R_i\frac{\phi_m^2}{\phi_h} \tag{4.32}$$

Finally, this can be rearranged to yield:

$$R_i = \zeta\frac{\phi_h}{\phi_m^2} \tag{4.33}$$

The results are shown graphically in Figure 4.3.

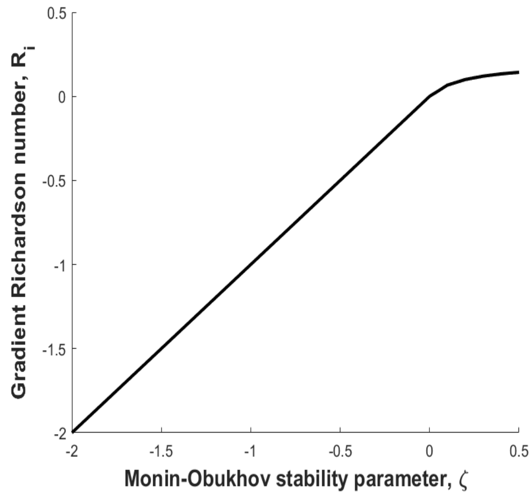

Figure 4.3: Relationship between gradient Richardson number and Monin-Obukhov stability parameter over the stability range -2 < ζ < 0.5.

4.17 Plot the turbulent Prandtl number, defined as the ratio of the eddy diffusivity for momentum (K_m) to the eddy diffusivity for heat (K_h), as a function of the Monin-Obukhov stability parameter (ζ) over the stability range $-2 < \zeta < 0.5$.

From Original Equation 3.65, we know:

$$K_m = \frac{kz_g u_*}{\phi_m} \text{ and } K_h = \frac{kz_g u_*}{\phi_h}$$

So

$$\frac{K_m}{K_h} = \frac{\frac{kz_g u_*}{\phi_m}}{\frac{kz_g u_*}{\phi_h}} = \frac{\phi_h}{\phi_m}$$

We can solve for ϕ_h and ϕ_m with Original Equations 4.35, 4.36, and 4.37 for $-2 \leq \zeta < 1$. For $-2 < \zeta < 0$:

$$\phi_m = (1 - 16\zeta)^{-1/4} \text{ and } \phi_h = (1 - 16\zeta)^{-1/2}$$

so

$$\frac{K_m}{K_h} = \frac{\phi_h}{\phi_m} = \frac{(1 - 16\zeta)^{-1/2}}{(1 - 16\zeta)^{-1/4}} = (1 - 16\zeta)^{-1/4}$$

For $0 \leq \zeta < 1$:

$$\phi_m = \phi_h = 1 + 5\zeta$$

so

$$\frac{K_m}{K_h} = \frac{\phi_h}{\phi_m} = 1$$

These results are shown graphically in Figure 4.4.

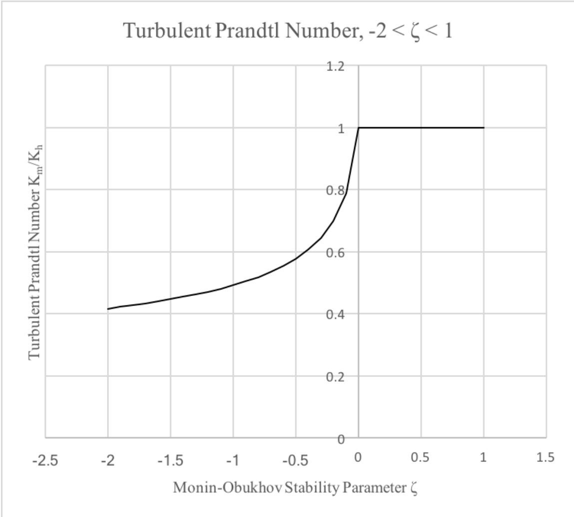

Figure 4.4: Prandtl number as a function of the Monin-Obukhov stability parameter ζ

4.18* In an observation in a potato farm, air temperature and wind speed are 19.86 °C and 1.46 m s^{-1} at the height of 1.57 m and 19.39 °C and 1.85 m s^{-1} at the height of 2.91 m above the ground. The displacement height is 0.70 m. (1) Use an iterative method to determine the Monin-Obukhov stability parameter (ζ), the momentum flux (F_m), and the eddy sensible heat flux (F_h) from the flux-gradient equations (Original Equations 3.61 and 3.62). In your calculation, you should use the geometric mean height (Original Equation 3.66) for ζ. (2) Calculate the gradient Richardson number and compare it with that predicted by Original Equation 4.47.

Given:

$T_1 = 19.86°\text{C}$ (293.01 K), $u_1 = 1.46$ m s^{-1}, $z_1 = 1.57$ m

$T_2 = 19.39°\text{C}$ (292.54 K), $u_2 = 1.85$ m s^{-1}, $z_2 = 2.91$ m

displacement height 0.7 m

First calculate the geometric mean of the two heights:

$$z_g = [(z_2 - d)(z_1 - d)]^{1/2} = [2.91 - 0.7)(1.57 - 0.7)]^{1/2} = 1.39 \text{ m}$$

Step 1: Make an initial guess for u_* and K_h, assuming $\phi_h = 1$:

$$u_* = kz_g \frac{\overline{u_2} - \overline{u_1}}{z_2 - z_1} = (0.4)(1.39)\frac{(1.85 - 1.46)}{(2.91 - 1.57)} = 0.16 \text{ m s}^{-1}$$

$$K_h = kz_g u_* = (0.4)(1.39)(0.16) = 0.089 \text{ m}^2\text{s}^{-1}$$

Step 2: Use Original Equation 3.62, with dry air mass density at 293.15 K $\rho_d = 1.2041$ kg m^{-3}:

$$F_h = -\overline{\rho_d} c_p K_h \frac{\overline{T_2} - \overline{T_1}}{z_2 - z_1} = -(1.2041)(1004)(0.089)\frac{292.54 - 293.01}{2.91 - 1.57} = 37.96 \text{ J m}^{-2} \text{ s}^{-1}$$

Step 3: Calculate ϕ_m, ϕ_h, K_m, and K_h (using $\overline{\theta} = 293$ k):

$$\zeta = -\frac{kz_g g}{u_\star{}^3 \overline{\theta}} \overline{w'\theta'} = -\frac{kz_g g}{u_\star{}^3 \overline{\theta}} (F_h / \overline{\rho_d} c_p) = -0.14$$

$$\phi_m = (1 - 16\zeta)^{-1/4} = 0.75$$

$$\phi_h = (1 = 16\zeta)^{-1/2} = 0.56$$

$$K_m = kz_g u_* / \phi_m = 0.119 \text{ m}^2 \text{ s}^{-1}$$

$$K_h = kz_g u_* / \phi_h = 0.159 \text{ m}^2 \text{ s}^{-1}$$

Step 4: Use Original Equation 3.61 to calculate F_m and 3.62 (above) recalculate F_h:

$$F_m = K_m \frac{\overline{u_2} - \overline{u_1}}{z_2 - z_1} = (0.119)\frac{1.85 - 1.46}{2.91 - 1.57} = 0.035 \text{ m}^2 \text{ s}^{-2}$$

$$F_h = -\overline{\rho_d} c_p K_h \frac{\overline{T_2} - \overline{T_1}}{z_2 - z_1} = -(1.2041)(1004)(0.159)\frac{292.54 - 293.01}{2.91 - 1.57} = 67.4 \text{ J m}^{-2} \text{ s}^{-1}$$

Use Original Equation 3.45 to calculate the new friction velocity:

$$u_* = (-\overline{u'w'})^{1/2} = F_m^{1/2} = 0.19 \text{ m s}^{-1}$$

Second iteration using steps 3 and 4: $\zeta = -0.152, \phi_m = 0.735, \phi_h = 0.539, K_m = 0.144, K_h = 0.196, F_m = 0.0419, u_* = 0.205, F_h = 83.0$

Third iteration: $\zeta = -0.150, \phi_m = 0.736, \phi_h = 0.540, K_m = 0.155, K_h = 0.210, F_m = 0.0450, u_* = 0.212, F_h = 89.0$

Fourth iteration: $\zeta = -0.145, \phi_m = 0.741,_h = 0.549, K_m = 0.159, K_h = 0.215, F_m = 0.0463, u_* = 0.215, F_h = 91.0$

Fifth iteration: $\zeta = -0.142, \phi_m = 0.744, \phi_h = 0.553, K_m = 0.161, K_h = 0.216, F_m = 0.0468, u_* = 0.216, F_h = 91.7$

The changes in u_* and F_h are very small between the last two iterations. We have convergence.

The gradient Richardson number in the surface layer is:

$$R_i = \frac{\frac{g}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z}}{(\frac{\partial \overline{u}}{\partial z})^2}$$

The potential temperature gradient can be approximated as:

$$\frac{\partial \overline{\theta}}{\partial z} = \frac{\partial \overline{T}}{\partial z} = \frac{292.54 - 293.01}{2.91 - 1.57} = -0.351 \text{ K m}^{-1}$$

The velocity gradient is:

$$\frac{\partial \overline{u}}{\partial z} = \frac{1.85 - 1.46}{2.91 - 1.57} = 0.291 \text{ s}^{-1}$$

Plugging in these gradient values in the R_i definition, we obtain $R_i = -0.139$. This value is nearly identical to the ζ value in the last iteration. Indeed, field experiments have shown that under unstable conditions $R_i \simeq \zeta$.

4.19 The following conditions are reported for an atmospheric surface layer: sensible heat flux $\rho_d c_p \overline{w'\theta'} = 350.2 \text{ W m}^{-2}$, momentum flux $-\overline{u'w'} = 0.12 \text{ m}^{-2} \text{ s}^{-2}$, vertical velocity gradient $\partial \overline{u}/\partial z = 0.03 \text{ s}^{-1}$, potential temperature $\overline{\theta} = 298.1$ K. Find (1) the buoyancy and the shear production of turbulent kinetic energy, and (2) the flux Richardson number.

Buoyancy of turbulent kinetic energy is:

$$\frac{g}{\overline{\theta}}\overline{w'\theta'} = \frac{9.81 \text{m s}^{-2}}{298.1 \text{ K}} \frac{350.2 \text{ W m}^{-2}}{(1.1839 \text{ kg m}^{-3})(1004 \text{ J kg}^{-1} \text{ K}^{-1})} = 0.009696 \text{ m}^2 \text{ s}^{-3}$$

In this calculation we used dry air mass density at 298.1 K $\rho_d = 1.1839 \text{ kg m}^{-3}$ and specific heat of air $c_p = 1004 \text{ J kg}^{-1} \text{ K}^{-1}$

Shear production of turbulent kinetic energy is:

$$-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\frac{\partial \overline{v}}{\partial z}$$

In the surface layer, this can be simplified to:

$$-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} = (0.12 \text{ m}^2 \text{ s}^{-2})(0.03 \text{ s}^{-1}) = -0.0036 \text{ m}^2 \text{ s}^{-3}$$

The flux Richardson number is:

$$R_f = \frac{\text{buoyancy production}}{\text{shear production}} = \frac{\frac{g}{\theta}\overline{w'\theta'}}{\overline{u'w'}\frac{\partial \overline{u}}{\partial z}} = \frac{0.009696}{-0.0036} = -2.69$$

4.20 (1) Find the momentum and the heat eddy diffusivity at the height of 3.5 m above the surface if the friction velocity u_* is 0.25 m s^{-1} and the Monin-Obukhov stability parameter ζ is -0.3. (2) Repeat the calculation but with a ζ value of 0.3. How does thermal stratification impact turbulent diffusion?

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(1) First calculate ϕ_m and ϕ_h using the equations for $-5 < \zeta < 0$.

$\phi_m = (1 - 16\zeta)^{-1/4} = 0.644$

$\phi_h = (1 - 16\zeta)^{-1/2} = 0.415$.

Then calculate K_m and K_h.

$K_m = \frac{kzu_*}{\phi_m} = \frac{(0.4)(3.5)(0.25)}{0.644}\mathrm{m^2 s^{-1}} = 0.54\ \mathrm{m^2\,s^{-1}}$

$K_h = \frac{kzu_*}{\phi_h} = \frac{(0.4)(3.5)(0.25)}{0.415}\mathrm{m^2 s^{-1}} = 0.84\ \mathrm{m^2\,s^{-1}}$.

(2) Now repeat the calculation using $\zeta = 0.3$.

First calculate ϕ_m and ϕ_h using the equations for $0 \leq \zeta < 1$.

$\phi_m = \phi_h = 1 + 5\zeta = 2.5$.

Then calculate K_m and K_h.

$K_m = \frac{kzu_*}{\phi_m} = \frac{(0.4)(3.5)(0.25)}{2.5}\mathrm{m^2 s^{-1}} = 0.14\ \mathrm{m^2\ s^{-1}}$

$K_h = \frac{kzu_*}{\phi_h} = \frac{(0.4)(3.5)(0.25)}{2.5}\mathrm{m^2 s^{-1}} = 0.14\ \mathrm{m^2\ s^{-1}}$.

There is more turbulent diffusion in thermally unstable conditions because both buoyancy and shear contribute to turbulence generation. Buoyancy destroys turbulence in stable, thermally stratified conditions.

Chapter 5

Flow in Plant Canopies

5.1 Estimate the plant area index from the vertical distribution of plant area density shown in Figure 5.1.

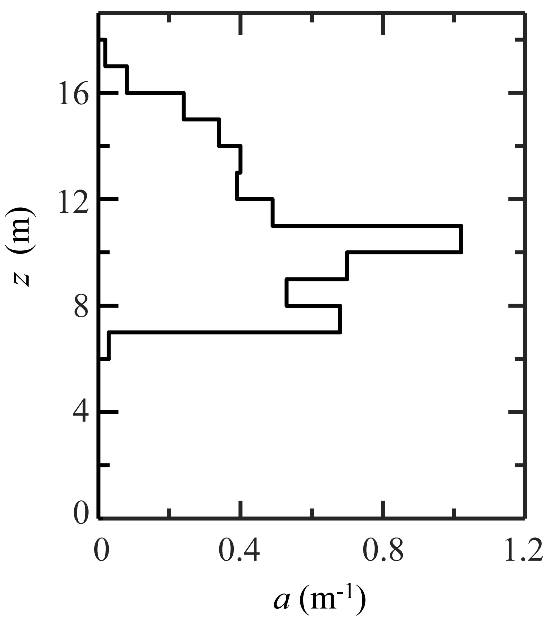


Figure 5.1: (Original Figure 5.12) Vertical distribution of plant area density measured in a Douglas-fir forest.

Plant area index $L = \int_0^h a\,dz = \sum_{i=0}^{N} a_i \times 1\,\mathrm{m} = (0.02 + 0.7 + 0.52 + 0.72 + 1.0 + 0.5 + 0.4 + 0.4 + 0.34 + 0.24 + 0.08 + 0.02) = 4.94.$

5.2 Are spatial differentiation and volume averaging commutable for Reynolds covariances, such as $\overline{u'w'}$ and $\overline{w'T'}$, in a plant canopy? Why or why not?

Spatial differentiation and volume averaging are commutable for Reynolds covariances $\overline{u'w'}$ and $\overline{w'\theta'}$ in a plant canopy. According to Slattery′s averaging theorem, spatial differentiation and volume averaging are commutable if and only if :

$$\frac{1}{Q}\sum\iint_{A_i}\overline{\phi}n_x dA = 0$$

Due to the non-slip boundary condition, velocity are zero at the plant surface and so are covariances. So the above surface integration of $\overline{u'w'}$ and $\overline{w'\theta'}$ is zero.

5.3 Explain why spatial differentiation and volume averaging are not commutable for the mean CO_2 mixing ratio, $\overline{s}_c$, but are commutable for the velocity and the mixing ratio product, $\overline{u}\,\overline{s}_c$.

According to Slattery's averaging theorem, spatial differentiation and volume averaging are only commutable if the variable is constant along each plant element surface. The CO_2 mixing ratio is not constant along a hypostomatous leaf as shown in Original Figure 5.4. However, the no-slip boundary condition requires that velocity vanishes at all plant element surfaces. Thus, the velocity and mixing ratio product $\overline{u}\,\overline{s_c}$ is constant (zero) along plant element surfaces. It follows then that spatial differentiation and volume averaging are not commutable for the CO_2 mixing ratio, $\overline{s}_c$, but are commutable for the velocity and the mixing ratio product $\overline{u}\,\overline{s_c}$.

5.4 Derive from Original Equation 3.16 the continuity equation for the spatial fluctuating velocities (Original Equation 5.13).

The incompressibility equation for mean velocities is:

$$\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}+\frac{\partial\overline{w}}{\partial z}=0$$

By decomposing it into volume mean and deviation, we have:

$$\frac{\partial([\overline{u}]+\overline{u}'')}{\partial x}+\frac{\partial([\overline{v}]+\overline{v}'')}{\partial y}+\frac{\partial([\overline{w}]+\overline{w}'')}{\partial z}=0$$
$$\frac{\partial[\overline{u}]}{\partial x}+\frac{\partial[\overline{v}]}{\partial y}+\frac{\partial[\overline{w}]}{\partial z}+\frac{\partial\overline{u}''}{\partial x}+\frac{\partial\overline{v}''}{\partial y}+\frac{\partial\overline{w}''}{\partial z}=0$$

By subtracting the volume mean incompressibility equation $\partial[\overline{u}]/\partial x+\partial[\overline{v}]/\partial y+\partial[\overline{w}]/\partial z=0$ from above, we have:

$$\frac{\partial\overline{u}''}{\partial x}+\frac{\partial\overline{v}''}{\partial y}+\frac{\partial\overline{w}''}{\partial z}=0$$

which is the incompressibility equation for volume deviation.

5.5 Show that under the canopy volume averaging scheme, the total kinetic energy consists of three parts:

$$[\overline{E}_T]=\mathrm{MKE}+\mathrm{DKE}+\mathrm{TKE},$$

where

$$\text{MKE} = \frac{1}{2}([\,\overline{u}\,]^2 + [\,\overline{v}\,]^2 + [\,\overline{w}\,]^2),$$

$$\text{DKE} = \frac{1}{2}([\,\overline{u}''\overline{u}''\,] + [\,\overline{v}''\overline{v}''\,] + [\,\overline{w}''\overline{w}''\,]),$$

and

$$\text{TKE} = \frac{1}{2}([\,\overline{u'^2}\,] + [\,\overline{v'^2}\,] + [\,\overline{w'^2}\,]).$$

The term DKE represents dispersive kinetic energy.

The canopy volume average of the total kinetic energy is:

$$\begin{aligned}
[\overline{E}_T] &= \left[\frac{1}{2}\overline{(u^2+v^2+w^2)}\right] \\
&= \frac{1}{2}\left[\overline{(\overline{u}+u')^2+(\overline{v}+v')^2+(\overline{w}+w')^2}\right] \\
&= \frac{1}{2}\left[\overline{\overline{u}^2}+\overline{2\overline{u}u'}+\overline{u'^2}+\overline{\overline{v}^2}+\overline{2\overline{v}v'}+\overline{v'^2}+\overline{\overline{w}^2}+\overline{2\overline{w}w'}+\overline{w'^2}\right] \\
&= \frac{1}{2}\left[\overline{u}^2+\overline{v}^2+\overline{w}^2\right]+\frac{1}{2}\left[\overline{u'^2}+\overline{v'^2}+\overline{w'^2}\right] \\
&= \frac{1}{2}\left[([\overline{u}]+\overline{u}'')^2+([\overline{v}]+\overline{v}'')^2+([\overline{w}]+\overline{w}'')^2\right]+\frac{1}{2}\left([\overline{u'^2}]+[\overline{v'^2}]+[\overline{w'^2}]\right) \\
&= \frac{1}{2}\left([[\overline{u}]^2]+[2[\overline{u}]\overline{u}'']+[\overline{u}''\overline{u}'']+[[\overline{v}]^2]+[2[\overline{v}]\overline{v}'']+[\overline{v}''\overline{v}'']+[[\overline{w}]^2]+[2[\overline{w}]\overline{w}'']+[\overline{w}''\overline{w}'']\right)+\text{TKE} \\
&= \frac{1}{2}\left([\overline{u}]^2+[\overline{v}]^2+[\overline{w}]^2\right)+\frac{1}{2}\left([\overline{u}''\overline{u}'']+[\overline{v}''\overline{v}'']+[\overline{w}''\overline{w}'']\right)+\text{TKE} \\
&= \text{MKE}+\text{DKE}+\text{TKE}
\end{aligned}$$

5.6* Verify that under the assumption of Original Equations 5.25 and 5.26, respectively, Original Equations 5.24 and 5.27 are solutions to Original Equation 5.23.

Part 1: Verification of Original Equation 5.24.

Original Equation 5.23:

$$K_m\frac{\partial^2\overline{u}}{\partial z}+\frac{\partial K_m}{\partial z}\frac{\partial\overline{u}}{\partial z}=C_d a\overline{u}^2$$

Original Equation 5.24:

$$\frac{\overline{u}(z)}{\overline{u}(h)}=\exp[\alpha_1(\frac{z}{h}-1)]$$

Original Equation 5.25:

$$K_m=l^2\frac{\partial\overline{u}}{\partial z}$$

Substituting the above equations into Original Equation 5.23, we get:

$$l^2\frac{\partial\overline{u}}{\partial z}\frac{\partial^2\overline{u}}{\partial z}+l^2\frac{\partial^2\overline{u}}{\partial z}\frac{\partial\overline{u}}{\partial z}=C_d a\overline{u}^2$$

which simplifies to:

$$2l^2\frac{\partial\overline{u}}{\partial z}\frac{\partial^2\overline{u}}{\partial z}=C_d a\overline{u}^2 \tag{5.1}$$

Then adding in the derivatives of 5.24, the left-hand side of Equation 5.1 is

$$2l^2(\overline{u}(h))^2\frac{\alpha_1^3}{h^3}\exp[2\alpha_1(\frac{z}{h}-1)] \tag{5.2}$$

Substituting Original Equation 5.24 in the right-hand side of Equation 5.1, we obtain

$$C_d a(\overline{u}(h))^2\exp\Big[2\alpha_1(\frac{z}{h}-1)\Big] \tag{5.3}$$

These two sides are equal by noting that:

$$\frac{2l^2\alpha_1^3}{h^3} = C_d a$$

or

$$\alpha_1 = \left[\frac{C_d a h^3}{2l^2}\right]^{1/3} \tag{5.4}$$

So Original Equation 5.24 is a solution to Original Equation 5.23 as long as α_1 is given by Equation 5.4.

Part 2: Verification of Original Equation 5.27.

The verification can be done in Matlab with the code attached below. First, we rewrite $K_m \propto \overline{u}$ as $K_m = t\bar{u}$, where t is a constant. If Original Equation 5.27 holds, it should make the two sides of Original Equation 5.23 equal. If we substitute $K_m = t\overline{u}$ into Original Equation 5.23, we get:

$$t\overline{u}\frac{\partial^2\overline{u}}{\partial z} + \frac{\partial(t\overline{u})}{\partial z}\frac{\partial\overline{u}}{\partial z} = C_d a\overline{u}^2$$
$$t\Big[\overline{u}\frac{\partial^2\overline{u}}{\partial z} + \frac{\partial\overline{u}}{\partial z}\frac{\partial\overline{u}}{\partial z}\Big] = C_d a\overline{u}^2$$

```
>> syms u_z z C_d a alpha_2 u_h h;
            % constructing symbolic variables.
>> u_z = u_h * ((sinh(alpha_2 * z / h)) / sinh(alpha_2))^(1/2);
            % inputting Eq. 5.27
>> dudz = diff(u_z, z, 1);
            % calculating primary partial differential.
>> dduddz = diff(u_z, z, 2);
            % calculating secondary partial differential.
>> t = (C_d * a * u_z^2) / (u_z * dduddz + dudz^2);
            % calculating value for t, and simplify below.
>> simplify(t)
ans = (2*C_d*a*h^2)/alpha_2^2
```

From the calculation above, we get

$$t \equiv \frac{2C_d a h^2}{\alpha_2^2}$$

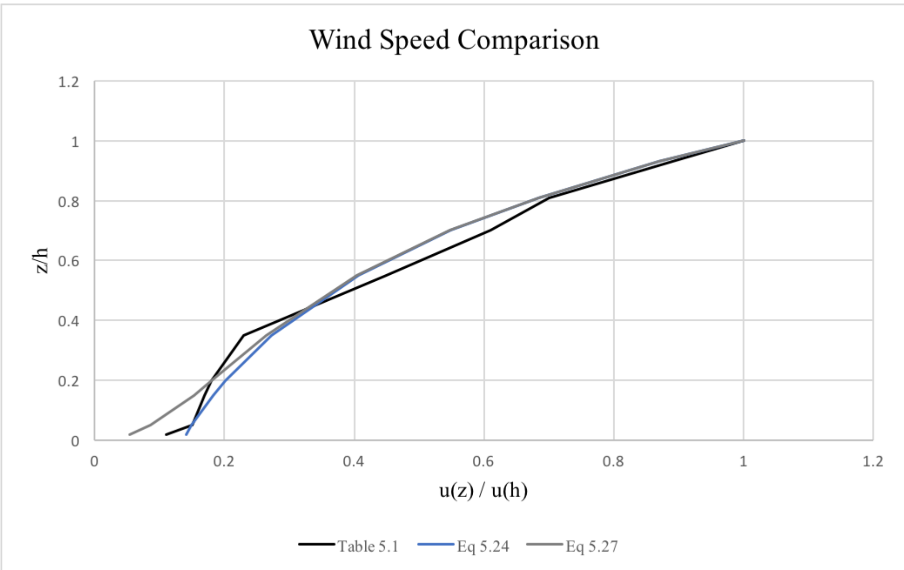
The constant t has the length dimension. Thus, with this constant, we have verified that under the assumption of Original Equation 5.26, Original Equation 5.27 is solution to Original Equation 5.23.

Table 5.1: (Original Table 5.1) Wind profile observed in a wind tunnel canopy made of uniform rods. Data source: Raupach and Thom (1981).

z/h	1.0	0.93	0.81	0.70	0.55	0.35	0.20	0.15	0.05	0.02
$\overline{u}(z)/\overline{u}(h)$	1.0	0.89	0.70	0.61	0.45	0.23	0.18	0.17	0.15	0.11

5.7 Compare in a profile plot the wind speed predicted by Original Equations 5.24 and 5.27 ($\alpha_1 = 2.0$ and $\alpha_2 = 4.0$) with the wind speed observed in a wind tunnel canopy (Original Table 5.1).

The results are given in the Figure below. The plot using Original Equation 5.27 rises more quickly than the plot using Original Equation 5.24 close to the ground, and they converge at larger heights. Original Equation 5.24 does not work as well close to the ground, but works in the middle and upper canopy. Original Equation 5.27 satisfies the no-slip condition at the ground surface where the wind speed vanishes. Both equations are empirical descriptions of the canopy wind profile and rarely satisfied by observational data. The observed wind shown in Original Table 5.1 is larger than calculated values near the ground and is good agreement with expected values at upper heights.



5.8 The canopy drag coefficient is 0.2, the wind profile is given by Original Equation 5.27 ($\alpha_2 = 4.4$), the plant area density is given in Original Figure 5.12, and the wind speed at the top of the canopy is 1.82 m s^{-1}. Calculate the canopy drag force at the height of $z/h = 0.5$.

Calculating the wind profile with Original Equation 5.27:

$$\frac{\overline{u}(z)}{\overline{u}(h)} = [\frac{\sinh(\alpha_2 z/h)}{\sinh \alpha_2}]^{1/2} \quad \text{or} \quad \overline{u}(z) = \overline{u}(h)[\frac{\sinh(\alpha_2 z/h)}{\sinh \alpha_2}]^{1/2}$$

$$\overline{u}(z) = (1.82 \text{ m s}^{-1})[\frac{\sinh((4.4)(0.5))}{\sinh 4.4}]^{1/2} = 0.60 \text{ m s}^{-1}$$

Since $z/h = 0.5$, looking at Original Figure 5.12, the plant canopy height $h = 18$ m, so $z = (0.5)(18) = 9$ m. At $z = 9$ m, the plant area density $a = 0.52 \text{ m}^{-1}$

Canopy drag force $= C_d a \overline{u}^2 = (0.2)(0.52 \text{ m}^{-1})(0.60 \text{ m s}^{-1})^2 = 0.037 \text{ m s}^{-2}$

5.9* A corn pollen grain dislodged from the top of the canopy is transported by the mean wind and falls at a settling velocity of 0.31 m s^{-1}. The canopy height is 2.2 m, wind speed at the canopy top is 2.3 m s^{-1}, and wind speed inside the canopy is described by Original Equation 5.24 with $\alpha_1 = 3.0$. How far does the pollen grain travel before it settles on the ground? (Assume that it will not be intercepted by plant elements.)

Wind profile in the canopy is described as:

$$\frac{\overline{u}(z)}{\overline{u}(h)} = \exp\left[\alpha_1(\frac{z}{h} - 1)\right]$$

where $\alpha_1 = 3.0$, $\overline{u}(h) = 2.3 \text{ m s}^{-1}$, and $h = 2.2$ m, so the wind profile is:

$$\overline{u}(z) = 2.3 \times \exp\left[3.0(\frac{z}{2.2} - 1)\right]$$

Let t be the pollen grain travel time. The grain's vertical position is:

$$z(t) = h - w_s t$$

where w_s is settling velocity. From this equation, the pollen lands on the ground ($z = 0$) at time:

$$\tau = h/w_s = 7.1 \text{ s}$$

The horizontal distance the pollen grain travels is determined by:

$$x = \int_0^{\tau} u(t)\, \mathrm{d}t = \int_0^{7.1} 2.3 \times \exp\left[3.0(\frac{2.2 - 0.31t}{2.2} - 1)\right] \mathrm{d}t$$

$$= 2.3 \int_0^{7.1} \exp(-0.42t)\mathrm{d}t = 5.2 \text{ m}$$

5.10* Some people interpret the displacement height, d, as the effective height of the mean canopy drag force, such that

$$d = \int_0^h z C_d a \overline{u}^2 \, \mathrm{d}z \,/ \int_0^h C_d a \overline{u}^2 \, \mathrm{d}z, \tag{5.5}$$

(Shaw and Pereira 1982). Use a numerical procedure to quantify d as a function of plant area index, L, according to Equation 5.5. The plant area density is given by

$$ah = \frac{L}{0.125\sqrt{2\pi}} \exp\left[-(z/h - 0.65)^2/(2 \times 0.125^2)\right], \tag{5.6}$$

where a has the dimensions of $\mathrm{m}^2\,\mathrm{m}^{-3}$, and the wind profile is given by Original Equation 5.24, with the wind extinction coefficient related to L ($0.5 < L < 7$) as $\alpha_1 = -0.0296L^2 + 0.6565L + 0.7010$.

Assume that the canopy height is 2.2 m, wind speed at the canopy top is 2.3 $\mathrm{m\,s^{-1}}$ and canopy drag coefficient is 0.2, we have:

$$\alpha_1 = -0.0296L^2 + 0.6565L + 0.7010$$

and

$$\begin{aligned}
&\overline{u}\,(z) = \overline{u}\,(h) \exp\left[\alpha_1 \left(\frac{z}{h} - 1\right)\right] \\
&= 2.3 \exp\left[\alpha_1 \left(\frac{z}{2.2} - 1\right)\right] \\
&= 2.3 \exp\left[\left(-0.0296L^2 + 0.6565L + 0.7010\right)\left(\frac{z}{2.2} - 1\right)\right]
\end{aligned}$$

$$\begin{aligned}
a &= \frac{L}{0.125h\sqrt{2\pi}} \exp\left[-(z/h - 0.65)^2/\left(2 \times 0.125^2\right)\right] \\
&= \frac{L}{0.125 \times 2.2\sqrt{2\pi}} \exp\left[-(z/2.2 - 0.65)^2/\left(2 \times 0.125^2\right)\right]
\end{aligned}$$

$$\begin{aligned}
d &= \int_0^h zC_d a\overline{u}(z)^2 dz \Big/ \int_0^h C_d a\overline{u}(z)^2 dz \\
&= \int_0^{2.2} 0.2za\overline{u}(z)^2 dz \Big/ \int_0^{2.2} 0.2a\overline{u}(z)^2 dz
\end{aligned}$$

This function can be evaluated in Matlab. The result is given in Figure 5.2. The result indicates that d is approximately 70 % of the canopy height.

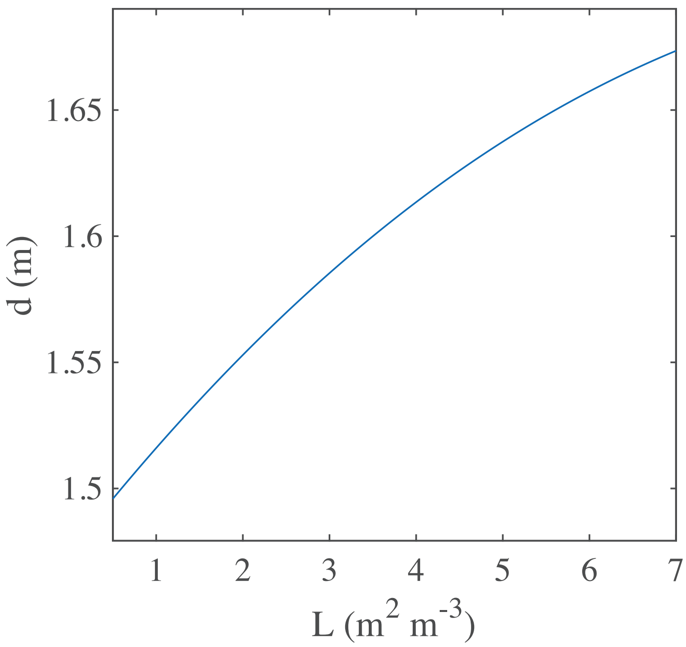


Figure 5.2: Diplacement height as a function of plant area index

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5.11 The plant area of a forest is evenly distributed between the ground and the top of the canopy, with the plant area index of 4.0. The mean wind speed at the top of the canopy is 1.82 $\mathrm{m\,s^{-1}}$ and the friction velocity is 0.42 $\mathrm{m\,s^{-1}}$. Assume that wind speed inside the canopy can be described by Original Equation 5.24 with $\alpha_1 = 2.4$. What is the canopy drag coefficient?

First we determine the plant area density a. Because the forest is evenly distributed, we have $a = L/h$. Next we calculate C_d from Original Equation 5.22:

$$\begin{aligned} u_*^2 &= \int_0^h C_d a \overline{u}^2 dz \\ &= C_d L(\overline{u}(h))^2 \int_0^h \exp[2\alpha_1(z/h-1)]d(z/h) \\ &= C_d L(\overline{u}(h))^2 \int_0^1 \exp[2\alpha_1(y-1)]dy \\ &= \frac{C_d L(\overline{u}(h))^2}{2\alpha_1}[1-\exp(-2\alpha_1)] \end{aligned}$$

$$C_d = \frac{2u_*^2\alpha_1}{L(\overline{u}(h))^2[1-\exp(-2\alpha_1)]} = 0.064$$

5.12 Wind load on an individual tree in a forest can be estimated as

$$\text{Wind load} = \rho C_d A \overline{u}_m^2,$$

where ρ is air density, A is total plant area of the tree, and $\overline{u}_m$ is wind speed at the middle point of its canopy. Calculate wind load on a tree with A of 80 m^2 distributed between heights $z = 0.5h$ and $z = 1h$ in a forest stand whose mean wind profile is given by Original Equation 5.24 with $\alpha_1 = 2.2$. Use a value of 0.2 for C_d, 1.20 $\mathrm{kg\,m^{-3}}$ for ρ and 2.6 $\mathrm{m\,s^{-1}}$ for $\overline{u}(h)$ for your calculation.

In this case, the mean height between $z = 0.5h$ and $z = 1h$ is $0.75h$. This height can be used with Original Equation 5.24 to find $\overline{u}(z)$, which will be the same as $\overline{u}_m$ in this case:

$$\frac{\overline{u}(z)}{\overline{u}(h)} = e^{\alpha(\frac{z}{h}-1)}$$

$$\overline{u}_m = \overline{u}(h)e^{\alpha_1(\frac{z}{h}-1)} = (2.6\ \mathrm{m\,s^{-1}})e^{2.2(0.75-1)} = 1.5\ \mathrm{m\,s^{-1}}$$

The equation for wind load can then be solved:

$$\text{wind load} = \rho C_d A \overline{u}_m^2 = (1.2\text{kg m}^{-3})(0.2)(80\ \mathrm{m}^2)(1.5\text{m s}^{-1})^2 = 43.2\ \mathrm{kg\,m\,s^{-2}} = 43.2\ \mathrm{N}$$

5.13 Estimate MKE production by the pressure gradient force and MKE destruction by canopy drag given $\partial\overline{p}/\partial x = 0.01\ \mathrm{hPa\,km^{-1}}$, $\overline{u} = 1.0\ \mathrm{m\,s^{-1}}$, $a = 0.4\ \mathrm{m^{-1}}$, and $C_d = 0.2$.

We use the first term in Original Equation 5.28 to calculate MKE production by pressure gradient force:

$$-\frac{\overline{u}}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x} = \frac{1.0}{1.20} \times 0.001\ \mathrm{Pa\,m^4 kg^{-1}\,s^{-1}} = 0.000833\ \mathrm{m^2\,s^{-3}}$$

We use the third term in Original Equation 5.28 to calculate MKE destruction by canopy drag:

$$-C_d a \overline{u}^3 = -0.2 \times 0.4\ \mathrm{m^{-1}} \times (1.0\ \mathrm{m\,s^{-1}})^3 = -0.08\ \mathrm{m^2\,s^{-3}}$$

The production term is several orders of magnitude smaller than the destruction term and can be safely omitted.

5.14 The plant area density is described by Equation 5.6, the plant area index is 3.0, the canopy drag coefficient is 0.2, the wind profile is given by Original Equation 2.7 ($\alpha_2 = 4.0$), and the wind speed at the top of the canopy is 2.5 $\mathrm{m\,s^{-1}}$. (1) Determine the momentum flux profile using Original Equation 5.21 for heights between the ground and the top of the canopy, (2) calculate shear production and wake production of TKE, and (3) present your results in a profile plot.

The momentum flux profile can be approximated by vertical summation,

$$-\overline{u'w'} = \int_0^z C_d a \overline{u}^2 dz' = \sum_{i=1}^{j} C_d a_i \overline{u}_i^2 (\Delta z)$$

where Δz is height increment, and $z = (\Delta z)j$.

The shear production profile is given as

$$-\overline{u'w'}\frac{\partial \overline{u}}{\partial z} = (-\overline{u'w'})_j (\frac{\partial \overline{u}}{\partial z})_j$$

where the vertical gradient of the velocity is found by differentiating Original Equation 5.27 with respect to z:

$$\frac{\partial \overline{u}}{\partial z} = \frac{\alpha_2 \overline{u}(h)}{2h[\sinh\alpha_2 \cosh(\alpha_2 \frac{z}{h})]^{1/2}}$$

The wake production at height z is:

$$C_d a \overline{u}^3 = C_d a_j \overline{u}_j^3$$

The Matlab code for computing these profiles is given below.

```
dz = 0.01; % height increment m
h=10; % canopy height, m
L=3; % plant area index
uh=2.5; % wind speed at the top of the canopy, m s-1
Cd=0.2; % canopy drag coefficient
n=round(10/dz);
z=[1:1:n]*dz;
% plant area density
a=(L/(0.125*(2*pi)^0.5*h))*exp(-(z/h-0.65).^2/(2*0.125^2));
% wind speed
u=uh*(sinh(4*z/h)/sinh(4)).^0.5;
% momentum flux profile
uw=ones(size(z))*0;
for j=2:n
    for i=1:j
      uw(j)=uw(j)+Cd.*a(i).*u(i).^2*dz;
    end;
end;
% wind speed gradient
dudz= (4*uh/2*h)./(sinh(4)*cosh(4*z/h)).^0.5;
%shear production
sp=uw.*dudz;
%wake production
wp=Cd*a.*u.^3;
plot(uw,z,'black--',sp,z,'black',wp,z,'r')
ylabel('height (m)')
xlabel('momentum flux, shear and wake production')
```

The final results are shown in Figure 5.3. In this calculation we assume the canopy height h = 10 m.

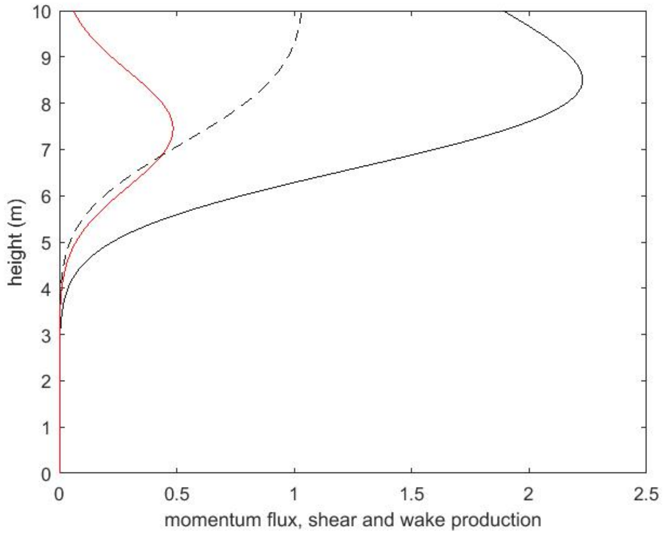


Figure 5.3: Profiles of momentum flux (dashed line, $\mathrm{m^2\,s^{-2}}$), shear production of TKE (solid black line, $\mathrm{m^2\,s^{-3}}$) and wake production of TKE (solid red line, $\mathrm{m^2\,s^{-3}}$).

5.15 Using the information provided in Problem 5.14 and Original Equation 5.30, (1) evaluate the transport, the shear destruction and the wake destruction term in the MKE budget equation, and (2) compare these terms in a profile plot.

Shear and wake destruction of MKE are equal to shear and wake production of TKE, and are balanced by the transport term according to Original Equation 5.30. The results are shown in Figure 5.4.

5.16 The wavenumber and the angular frequency of a wave event observed in a forest are 0.102 $\mathrm{rad\,m^{-1}}$ and 0.126 $\mathrm{rad\,s^{-1}}$, respectively. Calculate the wave speed, wavelength and wave period.

Given the wavenumber k and the angular frequency σ_r, the wave speed c_r, wavelength λ and the wave period T can be calculated as:

$$c_r = \frac{\sigma_r}{k} = \frac{0.126}{0.102} = 1.24 \text{ m s}^{-1}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.102} = 61.6 \text{ m}$$

$$T = \frac{2\pi}{\sigma_r} = \frac{2\pi}{0.126} = 49.9 \text{ s}$$

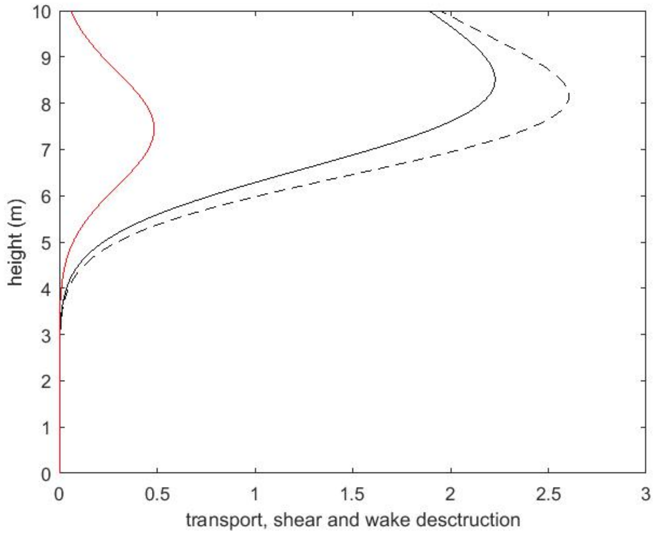


Figure 5.4: Profiles of MKE transport (dashed line, $\mathrm{m^2\,s^{-3}}$), shear destruction of MKE (solid black line, $\mathrm{m^2\,s^{-3}}$) and wake destruction of TKE (solid red line, $\mathrm{m^2\,s^{-3}}$).

5.17 Derive Original Equation 5.39 from Original Equations 5.37 and 5.38.

The vertical perturbation of the wave motion is given by Original Equation 5.36:

$$\tilde{w} = \hat{w}(z)\text{exp}[i(kx - \sigma t)]$$

Replacing the complex wave angular frequency with the summation of its real and imaginary parts ($\sigma = \sigma_r + i\sigma_i$):

$$\tilde{w} = \hat{w}(z)\text{exp}[i(kx - (\sigma_r + i\sigma_i)t)]$$
$$\tilde{w} = \hat{w}(z)\text{exp}[i(kx - \sigma_r t - i\sigma_i t)]$$
$$\tilde{w} = \hat{w}(z)\text{exp}[ikx - i\sigma_r t - i^2\sigma_i t]$$
$$\tilde{w} = \hat{w}(z)\text{exp}[ikx - i\sigma_r t + \sigma_i t]$$
$$\tilde{w} = |\hat{w}(z)|\text{exp}[i\phi_w(z)]\text{exp}[ikx - i\sigma_r t + \sigma_i t]$$
$$\tilde{w} = |\hat{w}(z)|\text{exp}(\sigma_i t)\text{exp}[i(kx - \sigma_r t + \phi_w(z))]$$
$$\tilde{w} = |\hat{w}(z)|\text{exp}(\sigma_i t)[\text{cos}(kx - \sigma_r t + \phi_w(z)) + i\text{sin}(kx - \sigma_r t + \phi_w(z))]$$

where we have used

$$\hat{w}(z) = |\hat{w}(z)|\,\text{exp}[i\phi_w(z)] \text{ (Original Equation 5.38)}$$

at the fifth step and the Euler's formula:

$$\text{exp}(i\theta) = \text{cos}\theta + i\text{sin}\theta$$

in the last step.

Keeping only the real terms of the solution, we get Original Equation 5.39:

$$Re\{\tilde{w}\} = |\hat{w}(z)|e^{\sigma_i t}\cos[kx - \sigma_r t + \phi_w(z)]$$

5.18 Provide an order-of-magnitude estimate for wavelength and period expected of the waves occurring in a 2-m tall corn canopy.

For a 2-m tall canopy, the fastest growing waves will become observable and will have a wavelength of:

$$\begin{aligned} \lambda &\approx 10h \text{ (Original Equation 5.53)} \\ &\approx 10 \times 2 = 20 \text{ m} \end{aligned}$$

and a speed of:

$$c_r \approx 1.6u_0(h) \text{ (Original Equation 5.54)}$$

where h is the canopy height and u_0 is the x-component of the horizontal velocity. The wave period, T, is given by:

$$T = \frac{2\pi}{\sigma_r} \text{ (Original Equation 5.40)}$$

and the angular frequency, σ_r, is given by:

$$\sigma_r = c_r k$$

where k is the wave number, which, for the fastest growing wave is:

$$k \approx \frac{0.6}{h}$$

Combining these equations, the wave period is:

$$T = \frac{2\pi h}{0.6 \times 1.6u_0(h)}$$

Assuming a light surface wind u_0 of 2.0 m s^{-1}, the wave period is on the order of 7 s.

5.19 The background state is specified as

$$N^2(z)/N^2(h) = (1 - \gamma_1)\exp[-\gamma_2(z/h - 1)] + \gamma_1, \tag{5.7}$$

and

$$u_0(z)/u_0(h) = \begin{cases} \exp[\alpha_2(z/h - 1)], & z/h \leq 1 \\ \alpha_1 \tanh[(\alpha_2/\alpha_1)(z/h - 1)] + 1, & z/h > 1 \end{cases} \tag{5.8}$$

where $N^2(h) = 0.002\ \mathrm{s}^{-2}$, $u_0(h) = 2.5\ \mathrm{m\,s}^{-1}$, $\alpha_1 = 3.0$, $\alpha_2 = 2.85$, $\gamma_1 = 0.2$ and $\gamma_2 = 2.0$, and h= 18.0 m. Determine the height of the inflection point and produce a profile plot of the gradient Richardson number for the air layer between $z = 0$ and $z = 3h$, where h is canopy height. Is shear instability likely to occur?

The vertical derivatives of the velocity are given as:

$$\frac{du_0}{dz} = \begin{cases} 0.40e^{2.85(\frac{z}{h}-1)} & 0 \le \frac{z}{h} \le 1 \\ 0.40[1 - \tanh^2(0.95(\frac{z}{h} - 1))] & 1 < \frac{z}{h} \le 3 \end{cases}.$$

$$\frac{d^2u_0}{dz^2} = \begin{cases} 0.063e^{2.85(\frac{z}{h}-1)} & 0 \le \frac{z}{h} \le 1 \\ 0.042[\tanh^3(0.95(\frac{z}{h} - 1)) - \tanh\big(0.95(\frac{z}{h} - 1)\big)] & 1 < \frac{z}{h} \le 3 \end{cases}.$$

As $\frac{d^2u_0}{dz^2}$ =0 at $z = h$ height, the height of the inflection is 18.0 m.

The gradient Richardson number R_i is given by:

$$R_i = \frac{N^2}{(\frac{du_0}{dz})^2} = \begin{cases} \frac{[0.8e^{2(\frac{z}{h}-1)}+0.2]4\times10^{-6}}{0.16e^{5.7(\frac{z}{h}-1)}} & 0 \le \frac{z}{h} \le 1 \\ \\ \frac{[0.8e^{2(\frac{z}{h}-1)}+0.2]4\times10^{-6}}{0.16[1-\tanh^2(0.95(\frac{z}{h}-1))]^2} & 1 < \frac{z}{h} \le 3 \end{cases}.$$

The minimum R_i is lower than 0.06 as shown in Figure 5.5, so shear instability will occur.

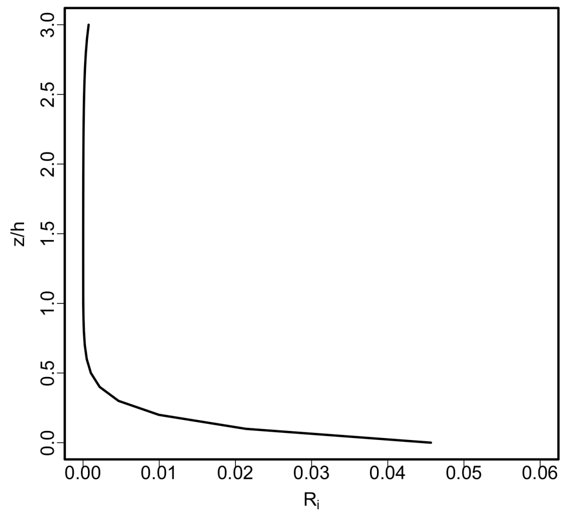


Figure 5.5: Profile of the gradient Richarson number

5.20 The growth rate of a wave observed in a forest is 0.0012 $\mathrm{rad\,s}^{-1}$. How long does it take for the wave amplitude to grow 2-fold, 10-fold and 100-fold?

The wave amplitude is given by:

$$A = |\hat{w}(z)|e^{\sigma_i t}$$

where t is the time and σ_i (=0.0012 rad s^{-1}) is the wave growth rate. If the wave amplitude grows twofold after time $T(=t + \delta t)$:

$$\begin{aligned}
2A &= |\hat{w}(z)|e^{\sigma_i T} \\
2|\hat{w}(z)|e^{\sigma_i t} &= |\hat{w}(z)|e^{\sigma_i (t+\delta t)} \\
2 &= e^{\sigma_i (\delta t)} \\
\delta t &= \frac{\ln 2}{\sigma_i} \\
&= \frac{\ln 2}{0.0012} \\
&= 577.6 \text{ s}
\end{aligned}$$

Similarly, time for the wave amplitude to grow tenfold is:

$$\delta t = \frac{\ln 10}{0.0012} = 1918.8 \text{ s}$$

and time for it to grow 100-fold is:

$$\delta t = \frac{\ln 100}{0.0012} = 3837.6 \text{ s}$$

Chapter 6

Balance of Forces in the Atmospheric Boundary Layer

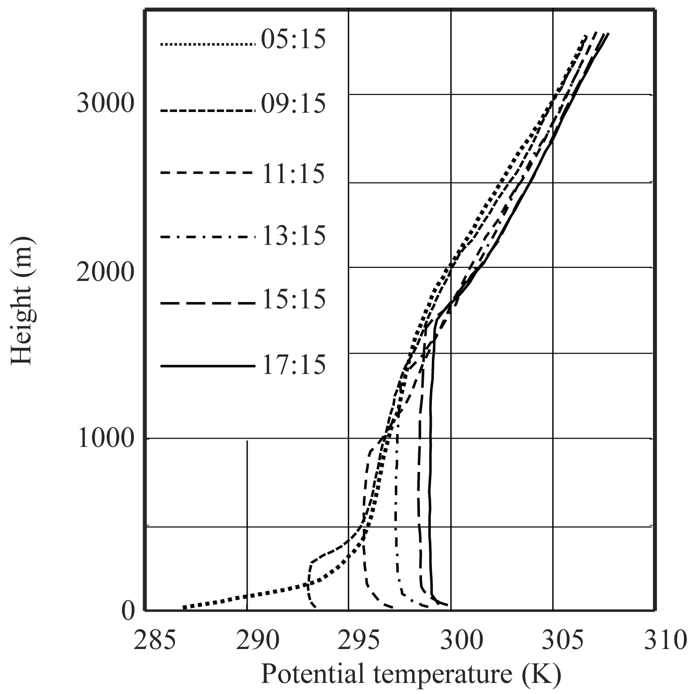


Figure 6.1: (Original Figure 6.11) Composite profiles of potential temperature in Saskatchewan, Canada in the summer of 1994. Time marks are local time. Data source: Barr and Betts (1997).

6.1 Figure 6.1 shows composite profiles of potential temperature observed in Saskatchewan, Canada in a summer season. Identify (1) the surface inversion layer and the residual layer at 06:15 local time, (2) determine the depth of the boundary layer at 09:15, 13:15, and 17:15, and (3) estimate the vertical potential temperature gradient in the free atmosphere.

---

(1) At 6:15, the sun will be rising so the surface inversion layer and residual layer will begin to weaken and be replaced by the mixed layer. It is possible that the two layers will still be

about what they were at 5:15am (0–200 m and 500–1000 m) but they will be completely gone by 9:15. (2) The boundary layer depth is 400 m at 09:15, 1500 m at 13:15, and 1750 m at 17:15. (3) The vertical potential temperature gradient is about 5 K km $^{-1}$ which is more than the average of 3.3 $\mathrm{K\,km^{-1}}$.

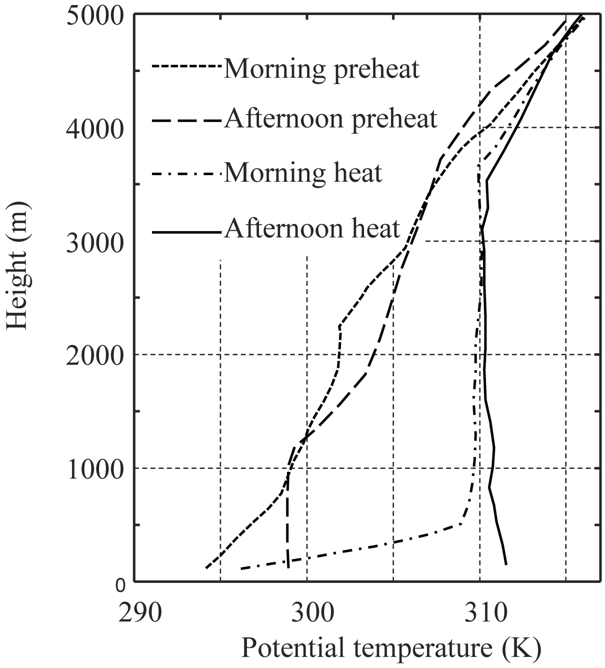


Figure 6.2: (Original Figure 6.12) Profiles of potential temperature before and during a mega heat-wave event in Voronezh, Russia. Data source: Miralles et al. (2014).

6.2 Figure 6.2 shows representative profiles of potential temperature before and during a heatwave event in Voronezh, Russia. Identify the surface inversion layer and the residual layer early in the morning and the "mixed layer" in the afternoon before and during the heatwave event.

Before the heat wave, the surface inversion layer and the residual layer in the morning are in 0–800 m and 800–2200 m, respectively. After the heat wave, they are in 0–500 m and 600–3700 m, respectively. The mixed layer before and after the heat wave in the afternoon is in 100–1200 m and 850–3550 m, respectively.

6.3 Estimate the depth of the viscous sub-layer using a typical value for surface friction velocity.

Take the Original Equation 6.1 as a reference, the thickness of viscous sublayer can be calculated as

$$\delta_l = \frac{5\nu}{u_*}$$

Under typical condition when u_* is approximately 0.3 $\mathrm{m\,s^{-1}}$, δ_l is

$$\delta_l = \frac{5 \times 1.48 \times 10^{-5}}{0.3} = 0.25 \text{ mm}$$

6.4 The horizontal pressure gradient is 0.01 hPa $\mathrm{km^{-1}}$. Find the geostrophic wind speed.

We note that 0.01 $\mathrm{hPa\,km^{-1}}$ = 0.001 $\mathrm{Pa\,m^{-1}}$ and that the mass density of air at STP $\overline{\rho}$ is 1.225 $\mathrm{kg\,m^{-3}}$. So the geostrophic wind speed is given as:

$$V_g = \frac{\partial \overline{p}/\partial x}{f\overline{\rho}} = \frac{0.001}{0.0001 \times 1.225} = 8.16 \text{ m s}^{-1}$$

6.5 The geostrophic wind speed is 6.0 $\mathrm{m\,s^{-1}}$. What is the horizontal pressure gradient?

$$\frac{\partial \overline{p}}{\partial x} = V_g f \overline{\rho} = 6.0 \text{ m s}^{-1} \times 0.0001 \text{ s}^{-1} \times 1.225 \text{ kg m}^{-3} = 0.735 \text{ Pa km}^{-1} = 0.00735 \text{ hPa km}^{-1}$$

6.6 Calculate the Ekman layer depth using three different values for the momentum eddy diffusivity (15, 1.0, and 0.3 $\mathrm{m^2\,s^{-1}}$).

Ekman layer depth can be calculated under the assumption that the momentum eddy diffusivity, K_m, does not change with height and using Original Equations 6.8 and 6.9:

$$z_i = \frac{\pi}{\gamma}$$

$$\gamma = (\frac{f}{2K_m})^{\frac{1}{2}}$$

These equations can be rearranged and combined to yield:

$$z_i = \frac{\pi}{(\frac{f}{2K_m})^{\frac{1}{2}}}$$

Recalling that the Coriolis parameter, f, is equal to 1×10^{-4} $\mathrm{s^{-1}}$, this equation can be solved for Ekman layer depth at each momentum eddy diffusivity value.

When K_m = 15 $\mathrm{m^2\,s^{-1}}$:

$$z_i = \frac{\pi}{(\frac{1\times10^{-4}}{2\times15})^{\frac{1}{2}}} = 1720 \text{ m}$$

When K_m = 1.0 $\mathrm{m^2\,s^{-1}}$:

$$z_i = \frac{\pi}{(\frac{1\times10^{-4}}{2\times1})^{\frac{1}{2}}} = 444 \text{ m}$$

When K_m = 0.3 $\mathrm{m^2\,s^{-1}}$:

$$z_i = \frac{\pi}{(\frac{1\times10^{-4}}{2\times0.3})^{\frac{1}{2}}} = 243 \text{ m}$$

6.7 The Ekman layer depth is 1200 m at midday and 300 m at midnight. What are the corresponding momentum eddy diffusivity values according to the Ekman spiral solution?

$$z_i = \frac{\pi}{\gamma}$$
$$\gamma = \frac{\pi}{z_i}$$
$$\gamma = (\frac{f}{2K_m})^{1/2}$$
$$K_m = \frac{f}{2\gamma^2}$$

With the Coriolis parameter $f = 1 \times 10^{-4}$ s^{-1}.

At an Ekman layer depth of 1200 m at midday:

$$\gamma = \frac{\pi}{1200 \text{ m}} = 0.00262 \text{ m}^{-1}$$
$$K_m = \frac{1 \times 10^{-4}\mathrm{s}^{-1}}{2(0.00262 \text{ m}^{-1})^2} = 7.3 \text{ m}^2\,\mathrm{s}^{-1}$$

At an Ekman layer depth of 300 m at night:

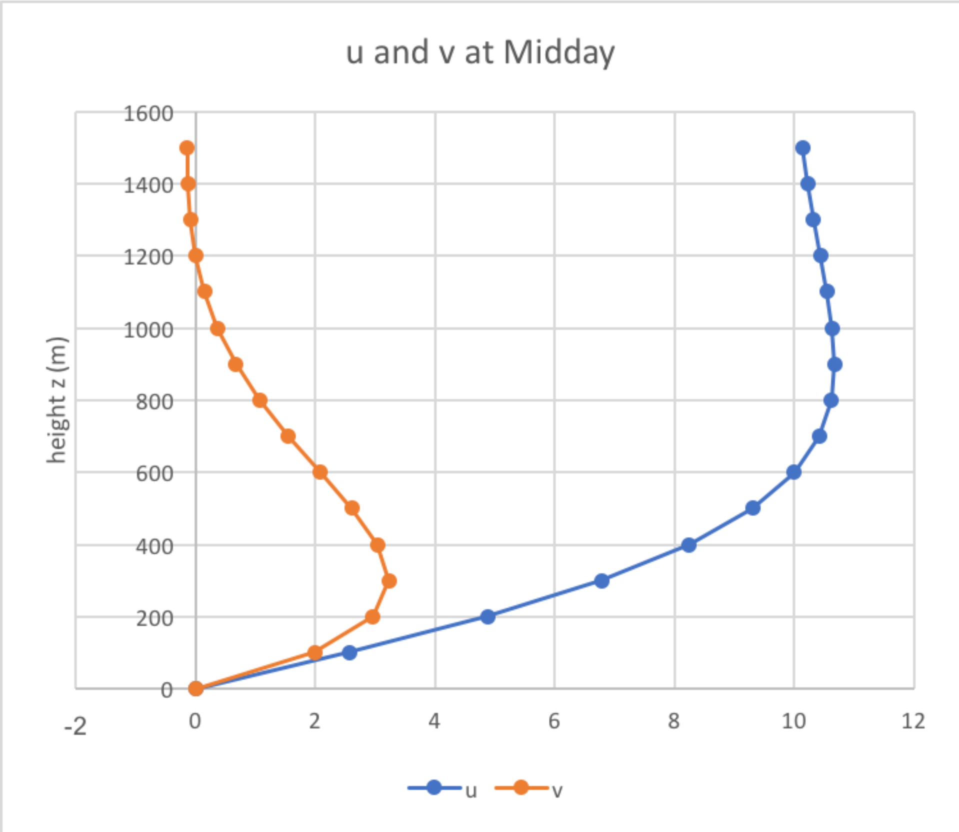
$$\gamma = \frac{\pi}{300 \text{ m}} = 0.0105 \text{ m}^{-1}$$
$$K_m = \frac{1\mathrm{x}10^{-4}\mathrm{s}^{-1}}{2(0.0105 \text{ m}^{-1})^2} = 0.46 \text{ m}^2\,\mathrm{s}^{-1}$$

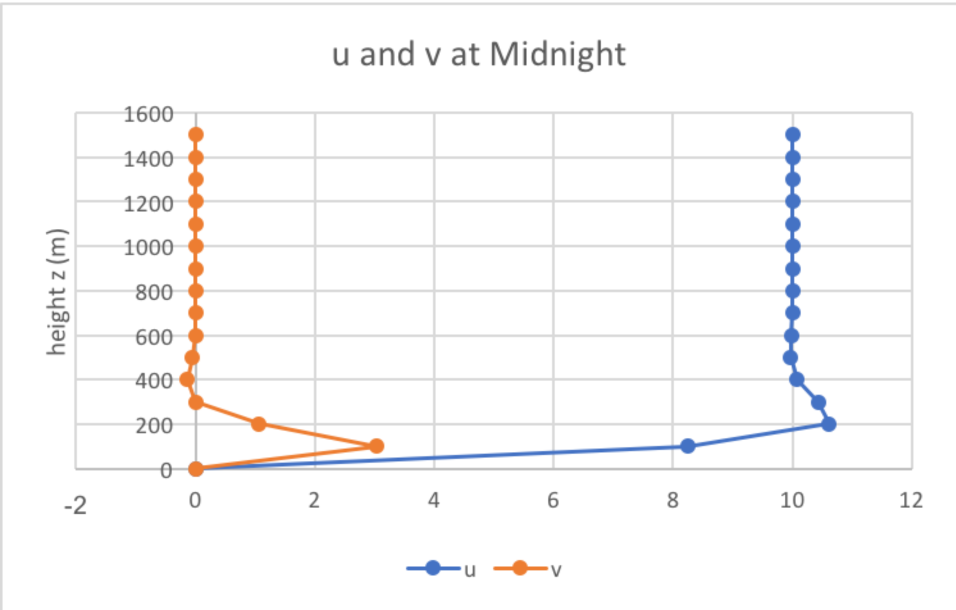
6.8 Using the eddy diffusivity values found in Problem 6.7, calculate the u and v velocity for the air layer between the ground and the 1500 m height. Present your results in a profile plot. (Hint: use the coordinate shown in Original Figure 4.1 and a geostrphic wind speed of 10 $\mathrm{m\,s^{-1}}$.)

From Original Equation 6.8:

$$\overline{u} = V_g(1 - e^{-\gamma z}\cos\gamma z)$$
$$\overline{v} = V_g(e^{-\gamma z}\sin\gamma z)$$

Using $\gamma = 0.00262$ m^{-1} at midday and $\gamma = 0.0105$ m^{-1} at midnight, you get the following profile plots for $\overline{u}$ and $\overline{v}$ from $z = 0$ m to $z = 1500$ m.





6.9* (1) Obtain a numerical solution to the momentum Equations 6.6 and 6.7 using the eddy diffusivity parameterization given by Original Equation 3.52, a geostrophic wind speed $V_g = 10.0\,\mathrm{m\,s^{-1}}$, a surface roughness $z_o = 1.0$ m, and a boundary layer depth $z_i = 1100$ m (Blackadar

1962). The surface and the upper boundary conditions are

$$\overline{u} = \overline{v} = 0 \text{ at } z = 0,$$

and

$$\overline{u} = V_g, \ \overline{v} = 0 \text{ at } z = z_i.$$

Air stabilty is neutral. Surface friction velocity is constrained by the Rossby similarity relation,

$$(u_*/V_g)^2 = k^2/\{[\ln(\mathrm{Ro}\, u_*/V_g) - A]^2 + B^2\}, \tag{6.1}$$

where Ro [$= V_g/(fz_o)$] is Rossby number, $A \simeq 2$, and $B \simeq 4.5$. Present your result in an Ekman spiral plot similar to Original Figure 6.5. (2) Repeat your calculation for $z_o = 0.001$ m, representing a smooth surface, such as a lake or the ocean, but with the other parameters unchanged. How does surface roughness affect the angle between wind direction near the surface and wind direction in the free atmosphere?

This solution uses Matlab solver pdepe for systems of partial differential equations. Because this solver is developed for intital and boundary condition problems, we will use the time dependent momentum equations (Original Equations 6.19 and 6.20). The initial profiles are approximated by linear interpolation between the values at the ground and at z_i. We find that the solution oscillates at initial time steps but eventually converges to a steady state. The solution at the last time step is taken as the steady-state solution or solution to Original Equations 6.6 and 6.7.

To be consistent with the Matlab notation, we switch to the following symbols:

$$\overline{u} \to u_1, \quad \overline{v} \to u_2, \quad z \to x$$

Rewrite Original Equations 6.19 and 6.20 in a matrix format required by the solver:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} .* \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{\partial}{\partial x} \begin{bmatrix} K_m(\partial u_1/\partial x) \\ K_m(\partial u_2/\partial x) \end{bmatrix} + \begin{bmatrix} f(u_2 - v_g) \\ -f(u_1 - u_g) \end{bmatrix}$$

where K_m is given by Orignal Equation 3.52.

The lower and upper boundary conditions are given as:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} .* \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 - u_g \\ u_2 - v_g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} .* \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The initial condition is approximated by linear interpolation between 0 and V_g:

$$u_0 = \begin{bmatrix} (x/z_i)u_g \\ (x/z_i)v_g \end{bmatrix}$$

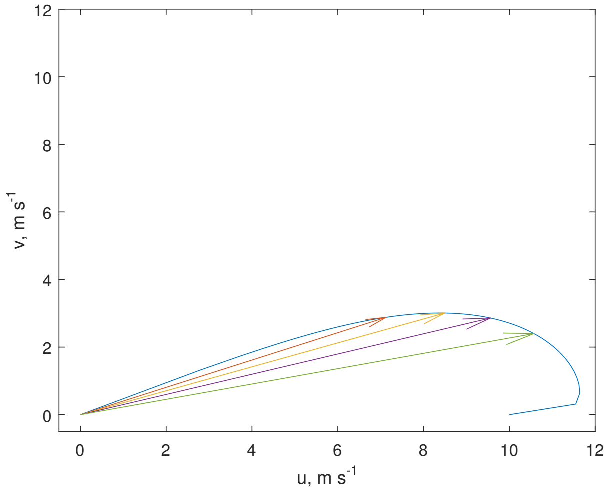


Figure 6.3: Ekman spiral solution for $z_0 = 1.0$ m. Arrows are wind vectors at heights $z/z_i = 0.2$, 0.4, 0.6 and 0.8.

We need to obtain the friction velocity u_* from Original Equation 6.23 first before starting the solution. This equation expresses u_* as an implicit function of the Rossby number. A graphic method of inverting this expression yields $u_* = 0.505\ \mathrm{m\,s^{-1}}$ for surface roughness $z_0 = 1.0$ m and $0.285\ \mathrm{m\,s^{-1}}$ for $z_0 = 0.001$ m. The Matlab code for solving these equations is given below.

The results (Figures 6.3 and 6.4) indicate much smaller wind rotation angles than given by the original Ekman solution. The angle between the surface wind (at the first grid height) and the geostrophic wind β_0 is 25.3° for $z_0 = 1.0$ m and reduces to 21.6° as the surface becomes smoother ($z_0 = 0.001$ m).

```
% partial differential equations
function [c,f,s] = pdex2pde(x,t,u,DuDx)
c = [1; 1];
ug=10; vg=0; % geostrophic wind component, m/s
ustar = 0.285; % surface friction velocity, 0.505 m/s at z0=1 m and 0.285 at ...
    z0=0.001 m
zi=1100; % height of the boundary layer,m
Km=0.4*x.*(1-x./zi).^2.*ustar;
f=Km.*DuDx;
s=0.0001.*[u(2)-vg;-(u(1)-ug)];
return;
% initial condition
function u0 = pdex2ic(x);
zi=1100; % boundary layer height
ug=10; vg=0; % geostrophic wind components, m/s
%linear interpolation as approximation for initial condition
u0 = [1; 1];
u0 =[(x./zi)*ug; (x./zi)*vg];
return;
%boundary conditions
function [pl,ql,pr,qr] = pdex2bc(xl,ul,xr,ur,t)
ug=10; vg=0; % geostrophic wind components, m/s
```

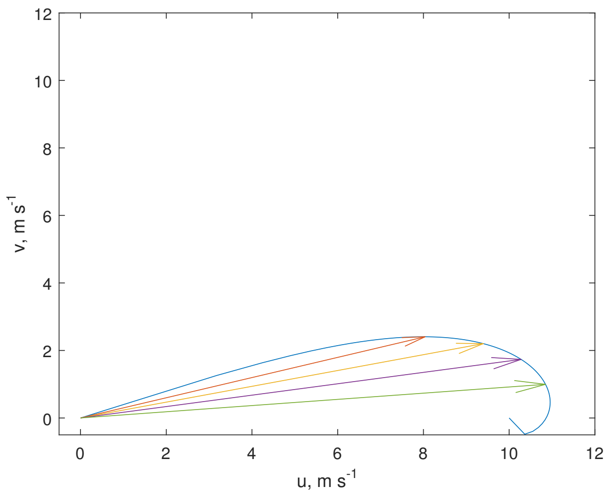


Figure 6.4: Ekman spiral solution for $z_0 = 0.001$ m. Arrows are wind vectors at heights $z/z_i = 0.2$, 0.4, 0.6 and 0.8.

```
pl = [ul(1); ul(2)];
ql = [0; 0];
pr = [ur(1)-ug; ur(2)-vg];
qr = [0; 0];
% numerical solution to problem 6.9
% determine ustar first from Equation 6.23, and replace the ustar value in
% the function pdex2pde
%
cd c:\x\lee\BLM_2017\ekman % folder where functions are stored;
zi=1100; % height of boundary layer, m
% select mess
x=[0:0.01:1]*zi; %vertical grid
t = [0:30:150*3600]; % time stepping every 30 s for 150 hours;
m = 0;
sol = pdepe(m,@pdex2pde,@pdex2ic,@pdex2bc,x,t);
%sol(i,j,k) approximates component k of the solution at time tspan(i) and the ...
    mesh point xmesh(j).
u = sol(:,:,1);
v = sol(:,:,2);
%surface wind angle, degree
beta_0= atan(v(length(t),2)/u(length(t),2))*180/pi;
% spiral plot
plot(u(length(t),:),v(length(t),:))
axis([-0.5 12 -0.5 12])
xlabel('u, m s^{-1}')
ylabel('v, m s^{-1}')
hold on
n=round(0.2*length(x));
quiver(0,0,u(length(t),n),v(length(t),n),0);
axis([-0.5 12 -0.5 12])
hold on
n=round(0.4*length(x));
quiver(0,0,u(length(t),n),v(length(t),n),0);
axis([-0.5 12 -0.5 12])
hold on
n=round(0.6*length(x));
quiver(0,0,u(length(t),n),v(length(t),n),0);
axis([-0.5 12 -0.5 12])
hold on
n=round(0.8*length(x));
quiver(0,0,u(length(t),n),v(length(t),n),0);
axis([-0.5 12 -0.5 12])
```

63 hold off

6.10 Estimate wind speed in the trunk space of a forest using the pressure gradient value found in Problem 6.5, a canopy drag coefficient of 0.2 and a plant area density of 0.05 $m^2\,m^{-3}$. What is the ratio of the wind speed in the trunk space to the geostrophic wind speed?

This problem can be solved using Original Equation 6.10:

$$0 = -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial x} - C_d a \overline{u} V \tag{6.2}$$

After the simplifying assumption that vertical motion is negligible and V is therefore equal to $\overline{u}$, we can simply plug the known values into Equation 6.10 to yield a wind speed of 0.245 $m\,s^{-1}$.

The geostrophic wind speed given in Problem 6.5 is 6 $m\,s^{-1}$.The wind speed ratio (wind speed in the trunk space to the geostrophic wind speed) is 0.04.

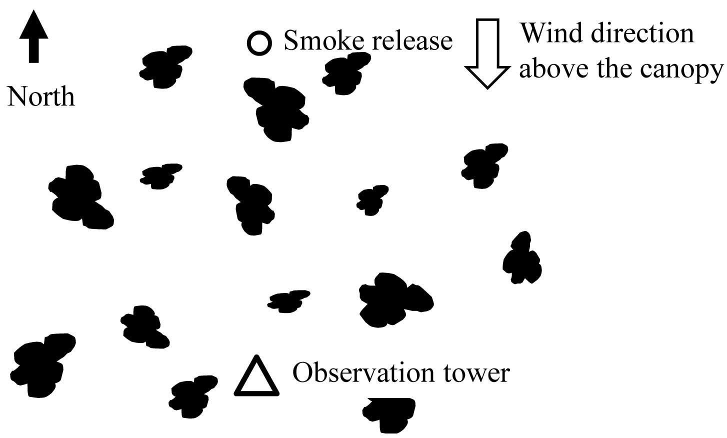


Figure 6.5: (Original Figure 6.13) A bird's eye view of a forest site showing tree crowns, a smoke source and an observational tower. Smoke is released on the ground at some distance away from the observational tower.

6.11 In a dispersion experiment in a forest, your instruments are mounted on a tower. Wind measurement above the forest indicates that air is moving from north, so you place a smoke source at some distance due north of the tower, hoping to observe the center of the smoke plume (Figure 6.5). Will the smoke plume follows a trajectory in the direction of the tower, or is it more likely to veer to the left or right of the tower? Why?

As it moves toward the observational tower, the smoke plume will veer to the left. In the Northern hemisphere, smoke released on the ground will move in the direction of pressure gradient force, but the air in the free atmosphere is moving in the direction of geostrophic wind. In between in the mixing layer, wind spirals counterclockwise as it gets closer to the surface. If measurements above the forest indicate air is moving from north to south, the smoke plume will move toward the east in the trunk layer, then spiral toward south as it moves into the surface atmosphere. At the observational tower, the direction of wind and smoke dispersal will be between east and south, so to the left of the tower (as viewed from the point of smoke release).

6.12 Derive Equations 6.13 and 6.14 from the full momentum Equations 3.31 and 3.32. What assumptions are made in your derivation?

Because the horizontal pressure gradient does not change with height, pressure in the residual layer can be presented by that in the free atmosphere. The full momentum equations:

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial t} &= -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} + f\overline{v} - \frac{\partial \overline{u'w'}}{\partial z} \\
&= \left[-\frac{1}{f}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x} + \overline{v} - \frac{1}{f}\frac{\partial \overline{u'w'}}{\partial z}\right] f \\
\frac{\partial \overline{v}}{\partial t} &= -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} - f\overline{u} - \frac{\partial \overline{v'w'}}{\partial z} \\
&= \left[-\frac{1}{f}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y} - \overline{u} - \frac{1}{f}\frac{\partial \overline{v'w'}}{\partial z}\right] f
\end{aligned}$$

will become:

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial t} &= \left[u_g + \overline{v} - \frac{1}{f}\frac{\partial \overline{u'w'}}{\partial z}\right] f \\
\frac{\partial \overline{v}}{\partial t} &= \left[v_g - \overline{u} - \frac{1}{f}\frac{\partial \overline{v'w'}}{\partial z}\right] f
\end{aligned}$$

where we have used the definition of geostrophic wind $u_g = -\frac{1}{f}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x}, v_g = \frac{1}{f}\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial y}$.

We also note that momentum fluxes in the residual layer equal zeros in stable conditions:

$$\frac{\partial \overline{u'w'}}{\partial z} = \frac{\partial \overline{v'w'}}{\partial z} = 0$$

Therefore, the above equations will become:

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial t} &= \frac{\partial}{\partial t}(\overline{u} - u_g) = f(\overline{v} - v_g) \\
\frac{\partial \overline{v}}{\partial t} &= \frac{\partial}{\partial t}(\overline{v} - v_g) = -f(\overline{u} - u_g)
\end{aligned}$$

6.13 Verify that Equations 6.15 and 6.16 are a solution to Equations 6.13 and 6.14 and satisfy the proper initial condition.

From Original Equation 6.13:

$$\frac{\partial}{\partial t}(\overline{u} - u_g) = f(\overline{v} - v_g)$$

Putting the solution from Original Equation 6.15 on the right-hand side of Original Equation 6.13, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}(\overline{u} - u_g) &= \frac{\partial((v_0 - v_g)\sin(ft) + (u_0 - u_g)\cos(ft))}{\partial t} \\
&= (v_0 - v_g)\frac{\partial(\sin(ft))}{\partial t} + (u_0 - u_g)\frac{\partial(\cos(ft))}{\partial t} \\
&= f(v_0 - v_g)\cos(ft) - f(u_0 - u_g)\sin(ft) \\
&= f(\overline{v} - v_g)
\end{aligned}$$

Similarly, from Original Equation 6.14:

$$\frac{\partial}{\partial t}(\overline{v} - v_g) = -f(\overline{u} - u_g)$$

Putting the solution from Original Equation 6.16 on the right-hand side of Original Equation 6.14:

$$\begin{aligned}
\frac{\partial}{\partial t}(\overline{v} - v_g) &= \frac{\partial((v_0 - v_g)\cos(ft) - (u_0 - u_g)\sin(ft))}{\partial t} \\
&= (v_0 - v_g)\frac{\partial(\cos(ft))}{\partial t} - (u_0 - u_g)\frac{\partial(\sin(ft))}{\partial t} \\
&= -f(v_0 - v_g)\sin(ft)) - f((u_0 - u_g)\cos(ft) \\
&= -f(\overline{u} - u_g)
\end{aligned}$$

At $t = 0$, from Original Equation 6.15:

$$\begin{aligned}
\overline{u} - u_g &= (v_0 - v_g)\sin(0) + (u_0 - u_g)\cos(0) \\
\overline{u} - u_g &= (u_0 - u_g) \\
\overline{u} &= u_0
\end{aligned}$$

Similarly, from Original Equation 6.16:

$$\begin{aligned}
\overline{v} - v_g &= (v_0 - v_g)\cos(0) - (u_0 - u_g)\sin(0) \\
\overline{v} - v_g &= (v_0 - v_g) \\
\overline{v} &= v_0
\end{aligned}$$

Thus, the solutions satisfy the proper initial conditions.

6.14 The initial velocity in the residual layer is 4.0 m s^{-1}, and the geostrophic velocity is 9.0 m s^{-1}. The angle between the two velocity vectors is 10.0°. Produce a vector plot of the inertial oscillation similar to the one shown in Figure 6.9. At about what time does the wind becomes super geostrophic? What is the highest wind speed expected?

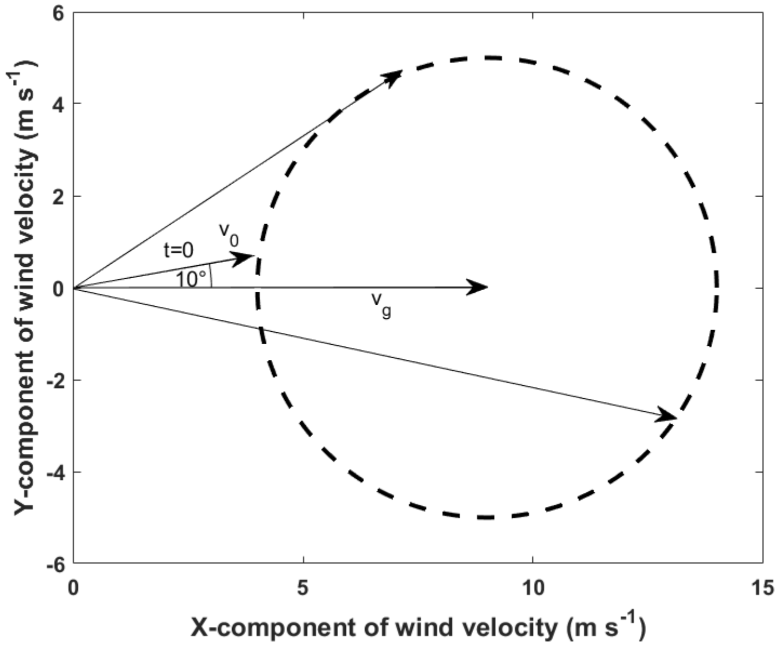


Figure 6.6: Vector plot of inertial oscillation

The vector plot of the inertial oscillation is given by Figure 6.6. The radius of the circle is given by Original Equation 6.17. Thus:

$$\begin{aligned} V_r &= [(9 - 4\cos 10)^2 + (4\sin 10)^2]^{1/2} \\ &= 5.11 \text{ m s}^{-1} \end{aligned}$$

The wind is super geostrophic when the wind overshoots beyond the speed of the geostrophic wind. This is roughly equal to:

$$\frac{\pi}{2f} \approx 4.5 \text{ hrs}$$

The highest expected wind speed is $V_g + V_r = 9 + 5.11 = 14.11$ m s^{-1}

6.15* The inertial oscillation starts at time $t = 0$ with a momentum eddy diffusivity of 12.5 m^2 s^{-1}, and the eddy diffusivity in the new equilibrium state is 0.5 m^2 s^{-1}. The Coriolis parameter

$f = 1.14 \times 10^{-4}$ s^{-1} (latitude 52°N). Produce a profile of the magnitude of the velocity in the boundary layer for $t = 0$ hr, 2 hr 11 min, 4 hr 23 min, 6 hr 35 min, and 8 hr 36 min. At what time(s) do you expect the occurrence of a low-level jet?

Let us assume that the mean geostrophic wind speed is 10 m s^{-1}. Thus, the x-component of the geosptrophic wind speed, u_g, is 10 m s^{-1} and the y-component, v_g, is 0 m s^{-1}. We assume that the momentum eddy diffusivity is constant with height, but changes with time. We find the initial velocity at each height of the boundary layer using the eddy diffusivity (K_m) of 12.5 m^2 s^{-1} and a Coriolis parameter (f) of 1.14×10^{-4} s^{-1} in the *Ekman spiral* solution (Original Equation 6.8). Similarly, we find the equilibrium velocity at each height using the *Ekman spiral* solution and an eddy diffusivity of 0.5 m^2 s^{-1} Finally, we plug in the values for u_0, v_0, u_e, and v_e at each height into Original Equations 6.21 and 6.22.

The magnitude of the wind velocity is given by:

$$v = (\overline{u}^2 + \overline{v}^2)^{1/2}$$

Figure 6.7 shows the vertical profile of the magnitude of wind velocity for $t = 0$ hr, 2 hr 11 min, 4 hr 23 min, 6 hr 35 min, and 8 hr 36 min. The figure suggests that we can expect low-level jets 4 hr 23 min, 6 hr 35 min, and 8 hr 36 min after the inertial oscillation starts. It should be noted that the figure shows the vertical profile up to a height of approximately 294 m, which is the height of the boundary layer at the equilibrium state. The boundary layer height is 1471 m at the start of the oscillation and is somewhere in between for the other times. We also assume that the pressure gradient, and thus, the geostrophic wind speed, does not change during this time interval.

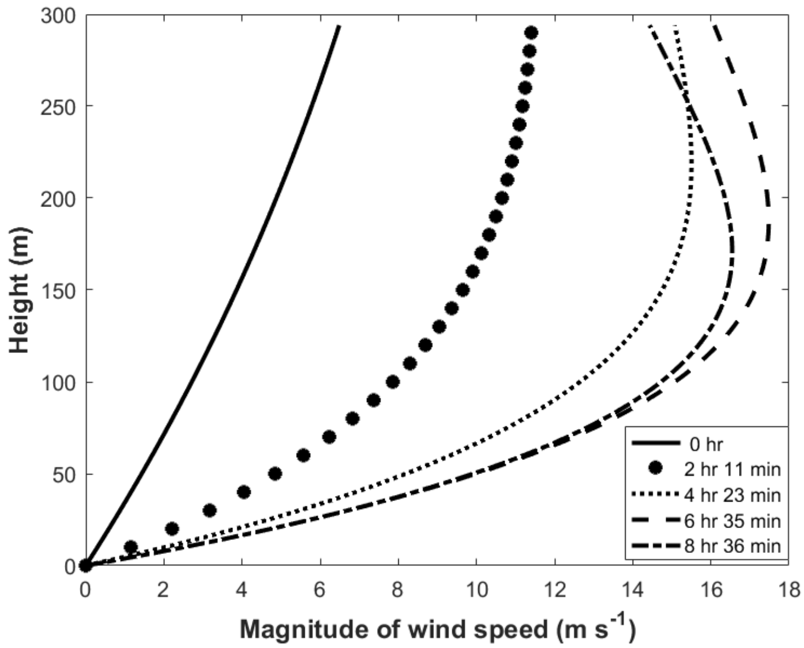


Figure 6.7: Vertical profile of nocturnal wind speed magnitude within the boundary layer at 52°N

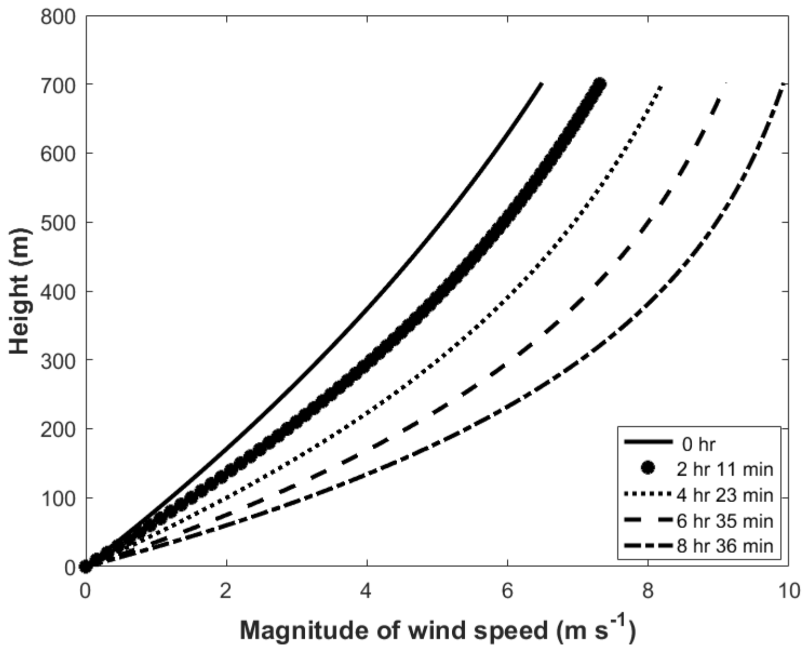


Figure 6.8: Vertical profile of nocturnal wind speed magnitude within the boundary layer at 8°N

6.16 Repeat the calculation in Problem 6.15 for a tropical latitude of 8°N. According to your result, is it possible for a low-level jet to establish at this latitude in the evening? Why or why not?

For 8°N, the Coriolis parameter, f, is given by:

$$f = 2\Omega \sin(\varphi) = 2 \times 7.2921 \times 10^{-5} \times \sin(8) = 0.20 \times 10^{-4}\ \mathrm{s}^{-1}$$

where Ω is the rotation rate of the Earth. Recalculating the magnitude of wind velocity as done in Problem 6.15 using this value, we obtain results shown in Figure 6.8. Figure 6.8 indicates that there will not be low-level jets at this latitude in the evening. This is because the oscillation period is too long to allow formation of the jets at night. The time period required for the wind to reach super geostrophic speeds is roughly 21 hrs, which is greater than the length of the night.

6.17 Can we use the Ekman spiral solution to describe wind profiles in a tropical boundary layer near the equator? Why or why not?

No. We can not use the Ekman spiral solution to describe wind profiles in a tropical boundary layer near the equator. As the Coriolis parameter, f, is very small near the equator, the force balance among pressure gradient, Coriolis force and turbulence flux divergence would no long hold and hence Ekman spiral is not a proper solution.

6.18 The wind is blowing from due south above the atmospheric boundary layer. Provide an estimate of the wind direction at the top of a forest and near the forest floor. (Hint: An open airspace exists between the ground and the forest canopy layer.)

As the wind in the free atmosphere is roughly geostrophic wind, southly wind suggests that pressure gradient is pointing westward, i.e., high pressure is on the east while low pressure on the west. At the top of the forest, Ekman spiral solution applies with $z = 0$, which states that wind would be at 45° angle with the geostrophic wind, deviating toward the low pressure, which is southeasterly. Near the forest floor, wind would blow along pressure gradient according to Original Equations 6.10 and 6.11, therefore wind would blow from the east.

6.19 Do you expect a larger wind directional shear in the boundary layer over an urban landscape or over a snow-covered pasture land? Why?

Approach 1:

We should expect a larger wind directional shear in the boundary layer over an urban landscape. Observational and numerical studies show that the angle decreases with decreasing surface roughness, approaching zero over the sea and large lakes. As the surface of a snow-covered pasture land is much smoother than the urban landscape, the angle at which the surface wind is deflected from the wind in free atmosphere is smaller, so the wind directional shear is smaller over a snow-covered pasture land.

Approach 2:

Larger wind directional shear would be expected in the boundary layer over a snow-covered pasture land. As a snow-covered pasture land would posses a smaller roughness than an urban landscape, it would also has smaller eddy diffusivity K_m, which would imply a shallower boundary layer. The wind directional shear within the boundary layer is fixed, which is of $\sim 45°$ according to the Ekman spiral solution, a smaller boundary layer height would imply a larger wind directional shear. Therefore, smoother snow-covered pasture land would have larger wind directional shear.

6.20* The geostrophic wind speed V_g is 10.00 $\mathrm{m\,s^{-1}}$, the initial velocity $\{u_0, v_0\}$ in the surface layer is $\{1.56, 1.34\}$ $\mathrm{m\,s^{-1}}$ and the equilibrium velocity $\{u_e, v_e\}$ is $\{5.53, 3.11\}$ $\mathrm{m\,s^{-1}}$. Find the velocity at successive time steps after sunset. Present your result in a vector form similar to those shown in Original Figure 6.9. Does your result suggest that the surface wind is directed toward higher pressure for some portion of the evening?

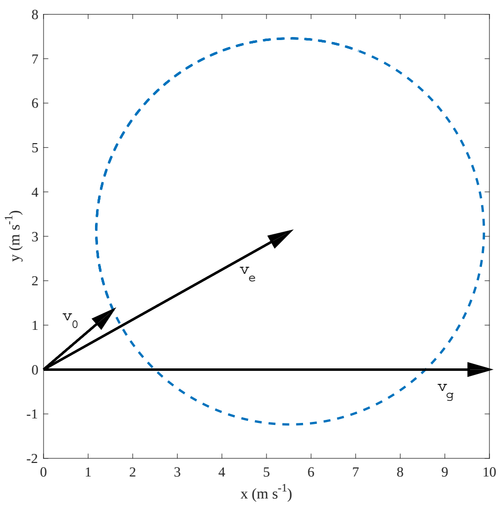


Figure 6.9: Evolution of velocity in the surface layer for Problem 6.20.

According to Original Equations 6.21 and 6.22, the trajectory of ($\overline{u}$,$\overline{v}$) is a circle around (u_e,v_e), with the radius decided by the distance between (u_0,v_0) and (u_e,v_e). The trajectory is clockwise around the circle shown in Figure 6.9, where f is taken to be 1.0×10^{-4}). As high pressure lies on the right of geostrophic wind, for the portion of the circle that lies below of x-axis, the wind is indeed directing toward the high pressure.

Chapter 7

Tracer Diffusion in the Lower Boundary Layer

7.1 Verify that the point-source solutions (Original Equations 7.12 and 7.13) satisfy global mass conservation (Original Equation 7.3).

Original Equation 7.12:

$$\overline{c}(x,y,z,t) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2})$$

Original Equation 7.13:

$$\overline{c}(x,y,z,t) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp(-\frac{(x-\overline{u}t)^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2})$$

Original Equation 7.3:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{c}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = Q$$

Plug in Original Equation 7.12 into the right-hand side of 7.3:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2})\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z =$$
$$\frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\frac{x^2}{2\sigma_x^2})\exp(-\frac{y^2}{2\sigma_y^2})\exp(-\frac{z^2}{2\sigma_z^2})\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z =$$
$$\frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\left(\sigma_x(\frac{\pi}{2})^{1/2}\mathrm{erf}(\frac{x}{\sigma_x 2^{1/2}})\right)\Bigg|_{x=-\infty}^{x=\infty}$$
$$\times\left(\sigma_y(\frac{\pi}{2})^{1/2}\mathrm{erf}(\frac{y}{\sigma_y 2^{1/2}})\right)\Bigg|_{y=-\infty}^{y=\infty}\left(\sigma_z(\frac{\pi}{2})^{1/2}\mathrm{erf}(\frac{z}{\sigma_z 2^{1/2}})\right)\Bigg|_{z=-\infty}^{z=\infty}$$

Noting that $\mathrm{erf}(\infty) = 1$ and $\mathrm{erf}(-\infty) = -1$, this equation simplifies to:

$$\frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\left(\sigma_x\sigma_y\sigma_z(\frac{\pi}{2})^{3/2}2^3\right) = Q$$

So the global mass conservation (Original Equation 7.3) is satisfied.

When Original Equation 7.13 is plugged into Original Equation 7.3, we replace the dummy variable x by $(x - \overline{u}t)$, and the left-hand side is also equal to Q. So mass conservation is satisfied.

7.2 An instantaneous point source at the origin releases 0.4 kg of tracer in a homogeneous flow field. A few seconds later, the tracer plume has spread in three dimensions, with the dispersion parameters σ_x = 5 m, σ_y = 5 m, and σ_z = 3 m. The mean flow velocity is zero. Plot the tracer concentration distribution as a function of z for $x = y = 0$ m and for x = 10 m and y = 0 m.

In a homogeneous flow field with a mean flow velocity of zero, tracer concentration as a function of time is given by Original Equation 7.12:

$$\overline{c}(x,y,z,t) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right) \tag{7.1}$$

Under the conditions listed, when $x = y = 0$ m, this becomes:

$$\overline{c}(x,y,z,t) = \frac{0.4}{(2\pi)^{3/2}(5)(5)(3)}\exp\left(-\frac{z^2}{2(3)^2}\right) \tag{7.2}$$

This is shown graphically in Figure 7.1

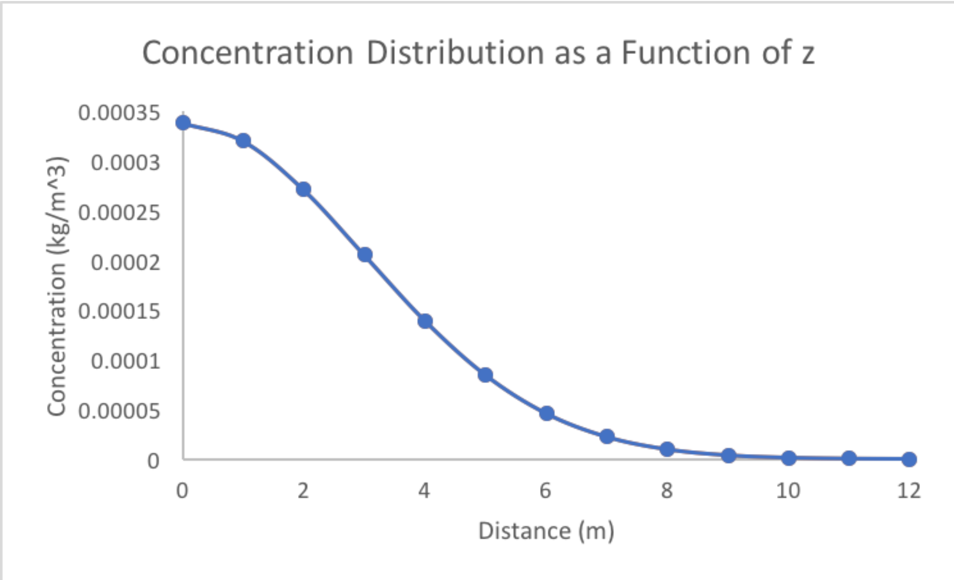


Figure 7.1: Tracer concentration distribution as a function of z for $x = y = 0$ m. At $z = 0$ m, concentration is 0.00034 $\mathrm{kg\,m^{-3}}$.

Under the conditions listed, when $x = 10$ m and $y = 0$ m, this becomes:

$$\overline{c}(x,y,z,t) = \frac{0.4}{(2\pi)^{3/2}(5)(5)(3)}\exp\left(-\frac{10^2}{2(5)^2} - \frac{z^2}{2(3)^2}\right) \tag{7.3}$$

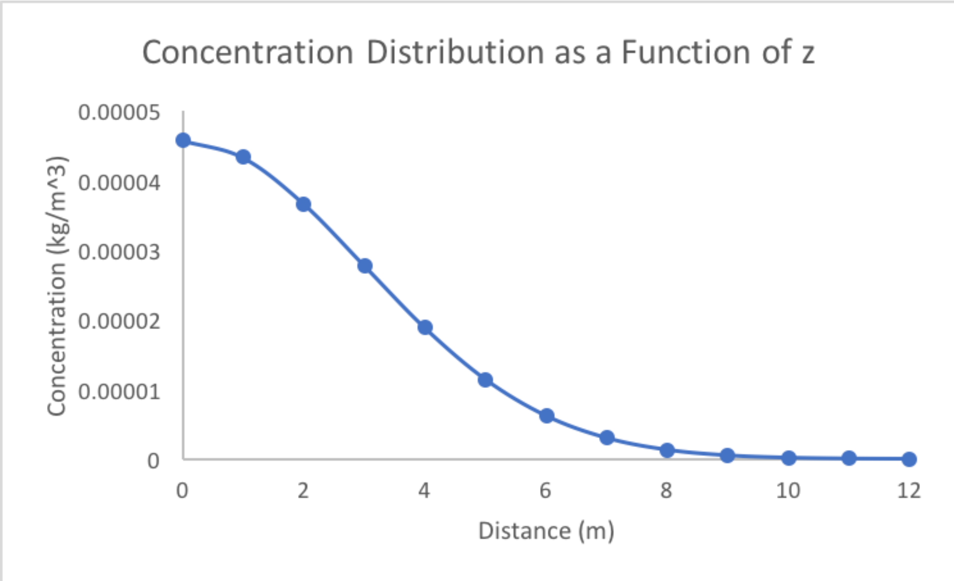


Figure 7.2: Tracer concentration distribution as a function of z for $x = 10$ m and $y = 0$ m. At $z = 0$ m, concentration is 4.6×10^{-5} kg m^{-3}.

The result is shown graphically in Figure 7.2.

7.3 The Lagrangian time scale (T_L) is 100 s and the vertical velocity standard deviation (σ_w) is 0.40 m s^{-1}. The flow field is homogeneous. Determine the vertical dispersion parameter (σ_z) at $t = 1$, 10, 50, 100, 200, 500 and 1000 s. Graph your result as a function of time.

The vertical dispersion parameter as a function of time can be found via Original Equation 7.19:

$$\sigma_z^2(t) = 2\overline{w_L^2}T_L^2\left[\frac{t}{T_L} - 1 + \exp\left(-\frac{t}{T_L}\right)\right]$$

Note that, in practice, the Langrangian velocity variance is approximated by the Eulerian velocity variance:

$$\overline{w_L^2} = \sigma_L^2$$

Substituting in this approximation and the given values, Original Equation 7.19 becomes

$$\sigma_z^2(t) = 2(0.4)^2(100)^2\left[\frac{t}{100} - 1 + \exp\left(-\frac{t}{100}\right)\right] \tag{7.4}$$

This can be solved at each time point listed (Figure 7.3).

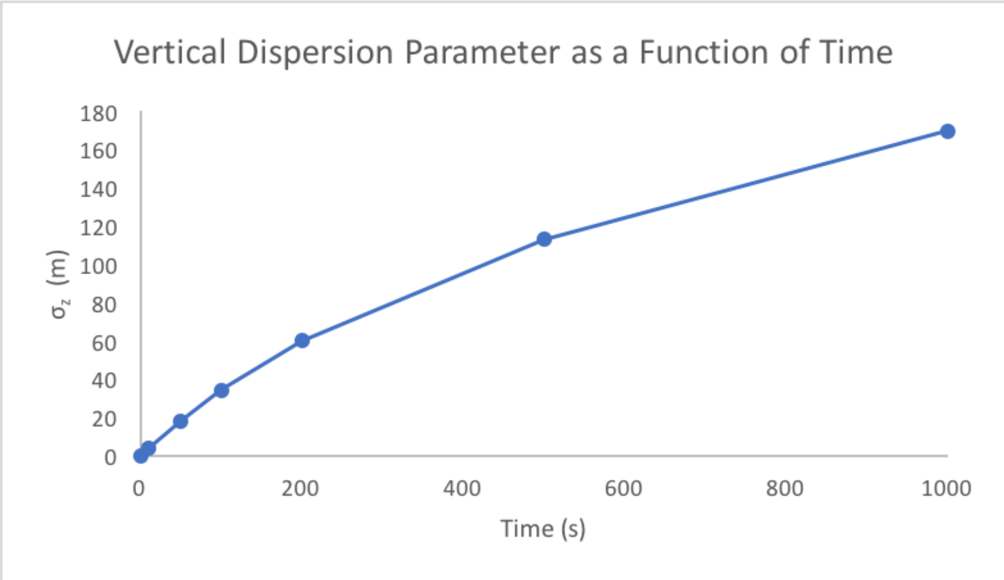


Figure 7.3: The vertical dispersion parameter increases with time as the plume spreads out.

7.4 A point source releases a puff of 1 g of SF_6 in a homogeneous flow field. The fluid velocity is 2.5 $\mathrm{m\,s^{-1}}$. Using the dispersion parameter values obtained in Problem 7.3, calculate the SF_6 concentration at a downwind distance of 250 m at the time steps indicated. Now repeat the calculation for zero wind speed. How does wind speed affect the tracer dispersion? (Assume that the turbulence is isotropic so that $\sigma_x = \sigma_y = \sigma_z$.)

---

(1) Fluid velocity $\overline{u} = 2.5$ m s^{-1} Original Equation 7.13:

$$\overline{c}(x,y,z,t) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp(-\frac{(x-\overline{u}t)^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2})$$
$$= \frac{1}{(2\pi)^{3/2}\sigma_z^3}\exp(-\frac{(250-2.5t)^2}{2\sigma_x^2} - \frac{0^2}{2\sigma_y^2} - \frac{0^2}{2\sigma_z^2})$$

The results are listed in the table below.

t (s)	σ_z (m)	$\overline{c}$ ($\mathrm{g\,m^{-3}}$)
1	0.399	0
10	3.93	0
50	18.5	1.12E-15
100	34.3	1.57E-6
200	60.3	5.33E-11
500	113	5.07E-25
1000	170	8.78E-47

(2) At wind speed $\overline{u} = 0$ m s^{-1} Original Equation 7.12:

$$\overline{c}(x,y,z,t) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2})$$
$$= \frac{1}{(2\pi)^{3/2}\sigma_z^3}\exp(-\frac{250^2}{2\sigma_x^2} - \frac{0^2}{2\sigma_y^2} - \frac{0^2}{2\sigma_z^2})$$

The results are listed in the table below.

t (s)	σ_z (m)	$\bar{c}$ (g m^{-3})
1	0.399	0
10	3.93	0
50	18.5	1.56E-45
100	34.3	4.65E-18
200	60.3	5.33E-11
500	113	3.82E-9
1000	170	4.39E-9

(3) How does wind speed affect tracer dispersion?

When there is mean wind speed, peak concentration will be reached earlier because the center of the plume moves at that speed and moves the tracer concentration.

Table 7.1: (Original Table 7.1) Coefficients of the Pasquill-Gifford empirical formulae (Seinfeld and Pandis 2006). Stability class: A – extremely unstable, B – moderately unstable, C – slightly unstable, D – neutral, E – slightly stable, F – moderately stable.

	A	B	C	D	E	F
A_y	−1.104	−1.634	−2.054	−2.555	−2.754	−3.143
B_y	0.9878	1.0350	1.0231	1.0423	1.0106	1.0148
C_y	−0.0076	−0.0096	−0.0076	−0.0087	−0.0064	−0.0070
A_z	4.679	-1.999	−2.341	−3.186	−3.783	−4.490
B_z	−1.7172	0.8752	0.9477	1.1737	1.3010	1.4024
C_z	0.2770	0.0136	−0.0020	−0.0316	−0.0450	−0.0540

7.5 The dispersion parameters (σ_y and σ_z) for the atmospheric boundary layer can be described by the Pasquill-Gifford empirical formulae,

$$\sigma_y(x) = \exp\left[A_y + B_y \ln x + C_y (\ln x)^2\right], \tag{7.5}$$

$$\sigma_z(x) = \exp\left[A_z + B_z \ln x + C_z (\ln x)^2\right], \tag{7.6}$$

where x is distance (in m) downwind from the smokestack. The empirical coefficients in these expressions have been determined experimentally for six stability classes (Original Table 7.1). Find the dispersion parameter values at downwind distances of 200 m and 2000 m for each of the stability classes. Does your result suggest that the smoke plume resembles a perfect cone shape?

Original Equations 7.76 and 7.77

$$\sigma_y(x) = \exp[A_y + B_y \text{ln}x + C_y(\text{ln}x)^2]$$

$$\sigma_z(x) = \exp[A_z + B_z \text{ln}x + C_z(\text{ln}x)^2]$$

At a downwind distance of 200 m:

Stability Class A, extremely unstable:

$$\sigma_y(x) = \exp[(-1.104) + (0.9878)\ln 200 + (-0.0076)(\ln 200)^2] = 50.2 \text{ m}$$

$$\sigma_z(x) = \exp[(4.679) + (-1.7172)\ln 200 + (0.2770)(\ln 200)^2] = 28.7 \text{ m}$$

Stability Class B, moderately unstable:

$$\sigma_y(x) = \exp[(-1.634) + (1.0350)\ln 200 + (-0.0096)(\ln 200)^2] = 35.9 \text{ m}$$

$$\sigma_z(x) = \exp[(-1.999) + (0.8752)\ln 200 + (0.0136)(\ln 200)^2] = 20.5 \text{ m}$$

Stability Class C, slightly unstable:

$$\sigma_y(x) = \exp[(-2.054) + (1.0231)\ln 200 + (-0.0076)(\ln 200)^2] = 23.4 \text{ m}$$

$$\sigma_z(x) = \exp[(-2.341) + (0.9477)\ln 200 + (-0.0020)(\ln 200)^2] = 13.8 \text{ m}$$

Stability Class D, neutral:

$$\sigma_y(x) = \exp[(-2.555) + (1.0423)\ln 200 + (-0.0087)(\ln 200)^2] = 15.2 \text{ m}$$

$$\sigma_z(x) = \exp[(-3.186) + (1.1737)\ln 200 + (-0.0316)(\ln 200)^2] = 8.5 \text{ m}$$

Stability Class E, slightly stable:

$$\sigma_y(x) = \exp[(-2.754) + (1.0106)\ln 200 + (-0.0064)(\ln 200)^2] = 11.3 \text{ m}$$

$$\sigma_z(x) = \exp[(-3.783) + (1.3010)\ln 200 + (-0.0450)(\ln 200)^2] = 6.3 \text{ m}$$

Stability Class F, moderately stable:

$$\sigma_y(x) = \exp[(-3.143) + (1.0148)\ln 200 + (-0.0070)(\ln 200)^2] = 7.7 \text{ m}$$

$$\sigma_z(x) = \exp[(-4.490) + (1.4024)\ln 200 + (-0.0540)(\ln 200)^2] = 4.2 \text{ m}$$

At a downwind distance of 2000 m, use the same equations but with a distance of 2000 m:

Stability Class A: $\sigma_y(x) = 389.6$ m and $\sigma_z(x) = 2059.2$ m

Stability Class B: $\sigma_y(x) = 292.5$ m and $\sigma_z(x) = 230.2$ m

Stability Class C: $\sigma_y(x) = 197.0$ m and $\sigma_z(x) = 115.2$ m

Stability Class D: $\sigma_y(x) = 129.6$ m and $\sigma_z(x) = 49.9$ m

Stability Class E: $\sigma_y(x) = 95.4$ m and $\sigma_z(x) = 33.3$ m

Stability Class F: $\sigma_y(x) = 64.5$ m and $\sigma_z(x) = 21.1$ m

No, the smoke plume does not represent a perfect cone shape, because the dispersion parameters in the y and z direction are not equal, and they spread out at different rates as the plume moves downstream from 200 to 2000 m. Typically the plume spreads more in the cross-wind direction than in the vertical direction.

7.6 A power plant emits SO_2 at the rate of 0.25 kg s^{-1} from a 50-m tall smokestack. The mean wind speed in the atmospheric boundary layer is 4.0 $m\,s^{-1}$. Determine the ground-level concentration of SO_2 below the center of the plume at distances of 50, 100, 200, 500 and 5000 m from the stack for an early morning hour (moderately stable, stability class F; Table 7.1) and a noon hour (extremely unstable; stability class A) and compare your estimates with the U. S. air quality standard. (The 1-hour U. S. national ambient air quality standard for SO_2 is 0.2 $mg\,m^{-3}$.) Responding to complaints from local residents about poor air quality in the area, the engineers plan to raise the stack height to 100 m. Will the increase in the stack height solve the existing air quality problems? Will it create new problems?

The given power plant should be regarded as an elevated point source with continuous emission. The concentration on the surface in nearby areas can be calculated with Original Equation 7.28:

$$\bar{c}(x, y, 0; z_1) = \frac{Q}{\pi \bar{u} \sigma_y \sigma_z} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) \exp\left(-\frac{z_1^2}{2\sigma_z^2}\right)$$

where Q, the rate of SO_2 emission, is 0.25 kg s^{-1}, and $\bar{u}$, the mean wind speed, is 4.0 m s^{-1}. The emission height, z_1, is 50 m.

In Original Equation 7.28, the dispersion parameters (σ_y and σ_z) are calculated with Pasquill-Gifford empirical formulae (Original Equations 7.76 and 7.77 given in Problem 7.5), where x is the distance between source and observation point, along the wind direction, and takes the values of 50, 100, 200, 500, and 5000 m and y=0.

For an early morning hour with moderately stable condition, stability class F from Original Table 7.1 should be used. The results of dispersion parameters (σ_y and σ_z) and concentrations at different distances are:

x (m)	50	100	200	500	5000
σ_y (m)	2.054	3.982	7.669	18.05	147.3
σ_z (m)	1.185	2.278	4.156	8.498	34.37
$\bar{c}(x, 0, 0; 50)$ (mg m^{-3})	0	0	0	0	1.364

During noon hour with extremely unstable condition, stability class A column should be used.

x (m)	50	100	200	500	5000
σ_y (m)	14.07	26.68	50.22	114.6	860.9
σ_z (m)	9.029	14.09	28.69	110.5	2.553×10^{4}
$\bar{c}(x, 0, 0; 50)$ (mg m^{-3})	3.435×10^{-5}	9.759×10^{-2}	3.025	1.418	9.052×10^{-4}

The surface air quality at 5000 m away from the power plant in early morning hour and at around 200 – 500 m from the plant at noon do not meet the US air quality standard.

If the stack is raised to 100 m, the value for z_1 in Original Equation 7.28 would be 100 m. The result for early morning hour is:

x (m)	50	100	200	500	5000
σ_y (m)	2.054	3.982	7.669	18.05	147.3
σ_z (m)	1.185	2.278	4.156	8.498	34.37
$\overline{c}(x,0,0;100)$ (mg m^{-3})	0	0	0	1.103×10^{-28}	5.705×10^{-2}

For noon hour, we have

x (m)	50	100	200	500	5000
σ_y (m)	14.07	26.68	50.22	114.6	860.9
σ_z (m)	9.029	14.09	28.69	110.5	2.553×10^{4}
$\overline{c}(x,0,0;100)$ (mg m^{-3})	3.626×10^{-25}	6.119×10^{-10}	3.183×10^{-2}	1.043	9.052×10^{-4}

The air quality is generally better with a taller smoke stack. However, the concentration of SO_2 still exceeds the standard of 0.2 mg m^{-3} at 500 m from the power plant at noon.

7.7 Show that the line source solution (Equation 7.32) satisfies both global and local mass conservation.

Original Equation 7.32:

$$\overline{c} = \frac{Q}{(\pi K_z \overline{u} x)^{1/2}} \exp\left(-\frac{\overline{u} z^2}{4 K_z x}\right)$$

The global conservation constraint for a ground-level line source is expressed as (Original Equation 7.30):

$$\int_0^\infty \overline{u}\,\overline{c}\,\mathrm{d}z = Q$$

To show that Original Equation 7.30 holds, we plug in Original Equation 7.32 to the right-hand side of Original Equation 7.30. Noting that in homogeneous turbulence, $\overline{u}$ and K_z are constant, we can introduce a new dummy variable:

$$r = \frac{z}{2}\sqrt{\frac{\overline{u}}{K_z x}}$$

The right-hand side of Original Equation 7.30 becomes:

$$\int_0^\infty \frac{\overline{u} Q}{(\pi K_z \overline{u} x)^{1/2}} \exp\left(-\frac{\overline{u} z^2}{4 K_z x}\right) \mathrm{d}z = Q \int_0^\infty \frac{2}{\pi^{1/2}} \exp(-r^2)\mathrm{d}r = \mathrm{erf}(\infty) Q = Q$$

So global mass conservation is satisfied.

Local conservation is expressed as (Original Equation 7.31):

$$\overline{u}\frac{\partial \overline{c}}{\partial x} = \frac{\partial}{\partial z}\left(K_z \frac{\partial \overline{c}}{\partial z}\right)$$

The derivative of c with respect to x is:

$$\frac{\partial \bar{c}}{\partial x} = \left[-\frac{Q}{2(\pi K_z \overline{u} x)^{1/2} x} + \frac{Q}{(\pi K_z \overline{u} x)^{1/2}} \frac{\overline{u} z^2}{4K_z x^2} \right] \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right)$$

The derivative of c with respect to z is:

$$\frac{\partial \bar{c}}{\partial z} = -\frac{Q}{2(\pi K_z \overline{u} x)^{1/2}} \frac{\overline{u} z}{K_z x} \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right)$$

Multiplying by K_z and differentiating with respect to z one more time, we have:

$$\frac{\partial}{\partial z}\left(K_z \frac{\partial \bar{c}}{\partial z} \right) = \overline{u} \left[-\frac{Q}{2(\pi K_z \overline{u} x)^{1/2} x} + \frac{Q}{(\pi K_z \overline{u} x)^{1/2}} \frac{\overline{u} z^2}{4K_z x^2} \right] \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right) = \overline{u} \frac{\partial \bar{c}}{\partial x}$$

Therefore, local mass conservation is satisfied.

7.8* Show that the mean plume height of a ground-level line source (a) is proportional to the square root of downwind distance in homogeneous turbulence, and (b) is proportional to downwind distance in a turbulent flow in which the mean velocity is constant with height but the eddy diffusivity increases linearly with height. Explain why the plume rises faster in the second flow configuration.

(a) According to Original Equations 7.41 and 7.32, K_z = constant and $\overline{u}$ = constant. We have

$$\begin{aligned}
\overline{Z} &= \int_0^\infty \bar{c} z dz \Big/ \int_0^\infty \bar{c} dz \\
&= \int_0^\infty \frac{Q}{(\pi K_z \overline{u} x)^{1/2}} \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right) z dz \Big/ \int_0^\infty \frac{Q}{(\pi K_z \overline{u} x)^{1/2}} \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right) dz \\
&= \int_0^\infty \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right) z dz \Big/ \int_0^\infty \exp\left(-\frac{\overline{u} z^2}{4K_z x} \right) dz \\
&= \left(2{K_z}^{1/2} x^{1/2} \right) \Big/ \left(\overline{u}^{1/2} \sqrt{\pi} \right)
\end{aligned} \tag{7.7}$$

Therefore, the mean plume height of a ground-level line source is proportional to the square root of downwind distance in homogeneous turbulence.

(b) According to Original Equations 7.41 and 7.32, $K_z = k u_* z$ and $\overline{u}$ = constant, we have

$$\begin{aligned}
\overline{Z} &= \int_0^\infty \bar{c} z dz \Big/ \int_0^\infty \bar{c} dz \\
&= \int_0^\infty \frac{Q}{k u_* x} \exp\left(-\frac{\overline{u} z}{k u_* x} \right) z dz \Big/ \int_0^\infty \frac{Q}{k u_* x} \exp\left(-\frac{\overline{u} z}{k u_* x} \right) dz \\
&= \frac{u_*}{\overline{u}} x
\end{aligned}$$

Therefore, the mean plume height of a ground-level line source is proportional to downwind distance in this turbulent flow.

The exact K_z value in Equation 7.7 is not known, but for the purpose of comparison with case b it is reasonable to approximate it as:

$$K_z = ku_*\overline{Z}$$

Combining this approximation with Equation 7.7 we have a new formulation for the mean plume height in homogeneous turbulence:

$$\overline{Z} = \frac{4ku_*}{\pi u}x \simeq 0.5\frac{u_*}{\overline{u}}x$$

Indeed the plume rises faster in case b than in case a. The faster upward rise in the inhomogeneous turbulence may be caused by the asymmetric nature of the diffusivity: diffusion above a given height is stronger than below this height, causing fluid particles to move upward faster than if the diffusion efficiency is constant.

7.9 Determine the mean plume height of a ground-level line source in the atmospheric surface layer for three surface roughness values: $z_o = 0.001$ m, 0.05 m and 1 m. The air stability is neutral. Graph your result as a function of downwind distance (from 1 m to 100 m). How does surface roughness affect the plume rise?

By solving Original Equation 7.43 numerically in Matlab, we can obtain mean plum heights $\overline{Z}$ for downwind distances x from 1 to 100 m, based on different surface roughness. The changes of mean plum heights as increasing downwind distances are showed as below.

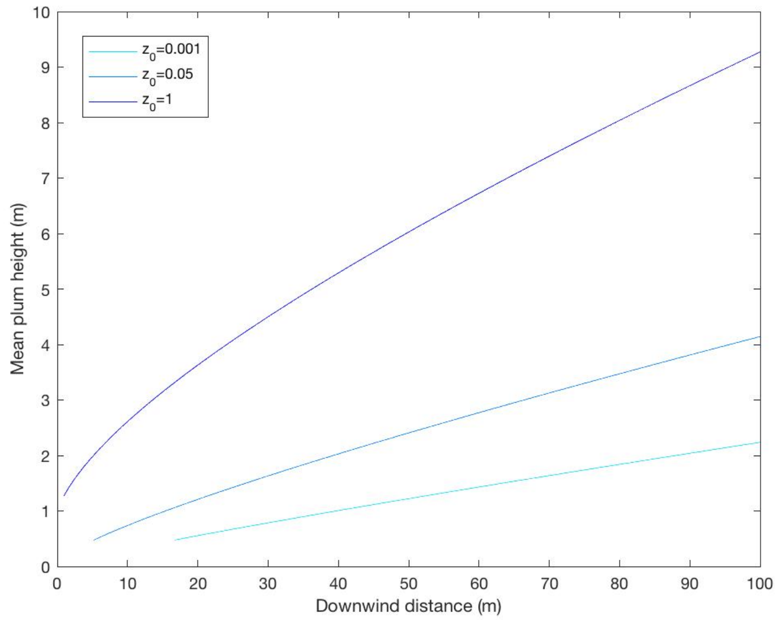


Figure 7.4: Mean plum height versus downwind distance for three surface roughness z_0 values (in m)

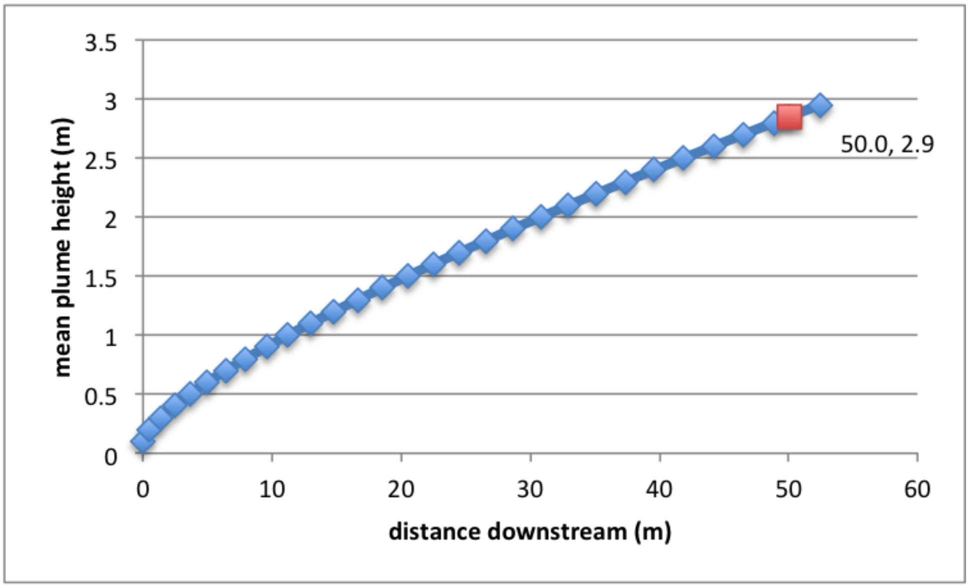
As illustrated in the figure, larger surface roughness enhances plume rise.

7.10 A ground-level line source emits a tracer at a rate of 0.20 $\mathrm{g\,m^{-1}\,s^{-1}}$ in the atmospheric surface layer. The friction velocity is 0.30 $\mathrm{m\,s^{-1}}$, the surface roughness is 0.1 m and air stability is neutral. Determine the mean plume height and the mean plume velocity at a distance of 50 m downwind of the source. Produce a profile plot of the tracer concentration at this location.

To solve for mean plume height, start with Original Equation 7.44 because we have neutral conditions:

$$x = \frac{\overline{Z}}{k^2}[\ln\big(p\overline{Z}/z_o\big) - 1] - \frac{z_o}{k^2}[\ln(p) - 1]$$

To solve for $\overline{Z}$ for a given x, we must invert the function. We will invert the function graphically. We use $k = 0.4$, $p = 1.55$, and $z_o = 0.1$ m. Graphically the inverted function is shown in this figure:



and the mean plume height is 2.85 m at 50 m downwind of the line source.

To calculate mean plume velocity, use the Original Equation 7.45 for neutral stability:

$$u_p = \frac{u_*}{k} \ln\left(0.6\overline{Z}/z_o\right)$$

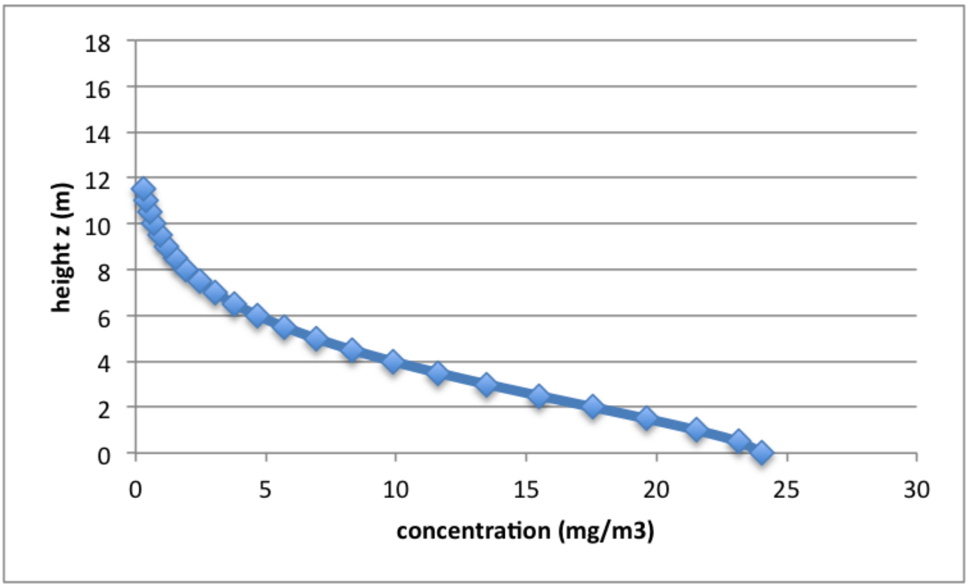
where $u_* = 0.30\ \mathrm{m\,s^{-1}}$, $k = 0.4$, $\overline{Z} = 2.85$ m, $z_o = 0.1$ m, and

$$u_p = \frac{0.30}{0.4} \ln(0.6 \times 2.85/0.1) = 2.13\ \mathrm{m\,s^{-1}}$$

Use Original Equation 7.40 to calculate tracer concentrations for the profile plot at 50 m downstream:

$$\overline{c}(x,z) = \frac{AQ}{\overline{Z}u_p} \exp\left[-(\frac{Bz}{\overline{Z}})^r\right]$$

where for neutral conditions: $r = 1.5$, $A = 0.73$, and $B = 0.66$. The resultant profile plot for $x = 50$ m is given in the figure below:



7.11 A tracer is released at the mid-canopy height in a plant canopy. The surface friction velocity is 0.30 $\mathrm{m\,s^{-1}}$, the canopy height is 20.0 m, and air stability is neutral. Estimate (a) the Lagrangian time scale for flow in the canopy, and (b) the eddy diffusivity at the source height. Roughly how far does the near field extend downwind of the source?

The Lagrangian time scale for flow is given by Original Equation 7.49:

$$T_L = \beta_1 h/u_* = 0.4 \times 20/0.3 = 26.67 \text{ s}$$

The source height is mid-canopy, or at $z = 10$ m. The eddy diffusivity is given by Original Equation 7.50:

$$K_z = \sigma_w^2 T_L = (\beta_2 u_* z/h)^2 T_L = (1.25 \times 0.3 \times 10/20)^2 \ = 0.94 \text{ m}^2\,\text{s}^{-1}$$

To determine the extent of the near field, we need the mean wind speed at the top of the canopy. Using Original Equation 3.47:

$$\overline{u} = \frac{u_*}{k} \ln \frac{z-d}{z_o} = \frac{0.3}{0.4} \ln \frac{20-14}{.2} = 2.55 \text{ m}\,\text{s}^{-1}$$

where we have used the approximate that the displacement height d and surface roughness z_o are about 0.7 and 0.1 of the canopy height, respectively. To find the mean wind speed at mid-canopy height, we use Original Equation 7.51:

$$\overline{u}(10) = 2.55(\frac{10}{20})^2 = 0.63 \text{ m}\,\text{s}^{-1}$$

Thus, the near field extends $\overline{u}(10) \times T_L = 0.64 \times 26.67 = 17$ m downwind of the source.

Alternatively, since only a rough estimate is required of the near field extent, let us take the mean velocity at source height $u(z_1)$ to be roughly 2.5 m s^{-1}. Then we can conclude that the near field extends roughly several tens of meters downwind of the source.

7.12 In a tracer dispersion experiment in a wind tunnel canopy, a line source at a height of 51 mm releases a tracer at a rate of 1.0 mg m^{-1} s^{-1}. The canopy height (h) is 60 mm. The friction velocity is 1.03 m s^{-1}, and the wind speed at the source height is 2.8 m s^{-1}. Predict the tracer concentration profile at downwind distances of $x/h = 0.38$, 1.32, 2.78, 5.72 and 11.6 from the source.

The concentration due to an elevated line source within a plant canopy is given by Original Equation 7.47:

$$\overline{c}(x, z; z_1) = \frac{Q}{\sqrt{2\pi}\sigma_z\,\overline{u}(z_1)}\Big\{\exp\Big[-\frac{(z-z_1)^2}{2\sigma_z^2}\Big] + \exp\Big[-\frac{(z+z_1)^2}{2\sigma_z^2}\Big]\Big\}$$

Given canopy height $h = 60$ mm, source height $z_1 = 51$ mm, rate of emission $Q = 1.0$ mg m^{-1} s^{-1}, friction velocity $u_* = 1.03$ m s^{-1}, and wind speed at source height $\overline{u}(z_1) = 2.8$ m s^{-1}, we obtain the Lagrangian time scale T_L:

$$T_L = \beta_1 h/u_* = 0.4 \times 0.06/1.03 = 0.023 \text{ s}$$

The vertical velocity standard deviation σ_w at source height is given by Original Equation 7.5:

$$\sigma_w(z_1) = \beta_2 u_* z_1/h = 1.25 \times 1.03 \times 51/60 = 1.09 \text{ m s}^{-1}$$

The square of the vertical dispersion parameter σ_z is given by Original Equation 7.48:

$$\begin{aligned}
\sigma_z^2(x; z_1) &= 2\sigma_w^2(z_1)\,T_L^2\left[\frac{x}{\overline{u}(z_1)T_L} - 1 + \exp\left(-\frac{x}{\overline{u}(z_1)T_L}\right)\right] \\
&= 2 \times (1.09)^2 \times (.023)^2 \times \left[\frac{x}{2.8 \times .023} - 1 + \exp\left(-\frac{x}{2.8 \times .023}\right)\right] \\
&= 0.0013 \times \left[\frac{x}{0.06} - 1 + \exp\left(-\frac{x}{0.06}\right)\right]
\end{aligned}$$

From the above equation, we obtain $\sigma_z = 0.0091$, 0.0276, 0.0489, 0.0784, and 0.1174 m s^{-1} at $x/h = 0.38$, 1.32, 2.78, 5.72 and 11.6 from the source, respectively.

Plugging these values into the original equation and plotting for $z = 0$ to 0.3 m, we get the concentration profiles (Figure 7.5).

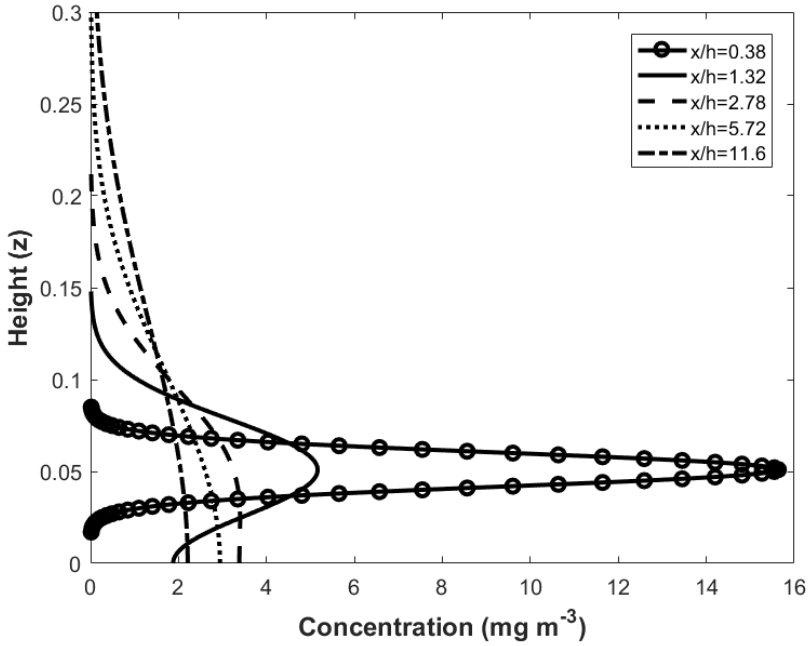


Figure 7.5: Tracer concentration profiles at different downwind distances from the emission source

Note: This problem is based on the wind tunnel study by Legg et al. (2000), *Boundary-Layer Meteorology*, 35: 277-302. Additional details can be found in Lee (2004), *Agricultural and Forest Meteorology*, 127: 131-141.

7.13* Derive an expression for the mean plume height of a tracer released from a ground-level line source inside a plant canopy. The wind speed and the eddy diffusivity are 2.3 $\mathrm{m\,s^{-1}}$ and 0.24 $\mathrm{m^2\,s^{-1}}$ at the top of the canopy, and the canopy height is 2.0 m. At what distance downwind of the source does the center of the plume ($\overline{Z}$) rise to the canopy top?

The main tool is Guassian integral, which states that:

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

and

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}$$

Employing Original Equation 7.53 for moderately dense canopy, we have:

$$\begin{aligned}
\int_0^\infty \bar{c}\,dz &= \frac{Qh^2\overline{u}_h^{1/2}}{2\sqrt{\pi}(xK_{z,h})^{3/2}}\int_0^\infty \exp\left(-\frac{\overline{u}_h}{4xK_{z,h}}z^2\right)dz \\
&= \frac{Qh^2\overline{u}_h^{1/2}}{2\sqrt{\pi}(xK_{z,h})^{3/2}}\cdot\frac{1}{2}\sqrt{\frac{4\pi xK_{z,h}}{\overline{u}_h}} \\
&= \frac{Qh^2}{xK_{z,h}}.
\end{aligned}$$

Changing variable $\xi = z^2$, we have:

$$\begin{aligned}
\int_0^\infty \bar{c} z\, dz &= \frac{Qh^2 \overline{u}_h^{1/2}}{2\sqrt{\pi}(xK_{z,h})^{3/2}} \int_0^\infty \exp\left(-\frac{\overline{u}_h}{4xK_{z,h}} z^2\right) z\, dz \\
&= \frac{Qh^2 \overline{u}_h^{1/2}}{2\sqrt{\pi}(xK_{z,h})^{3/2}} \int_0^\infty \frac{1}{2} \exp\left(-\frac{\overline{u}_h}{4xK_{z,h}} \xi\right) d\xi \\
&= \frac{Qh^2 \overline{u}_h^{1/2}}{2\sqrt{\pi}(xK_{z,h})^{3/2}} \cdot \left(\frac{4xK_{z,h}}{\overline{u}_h}\right) \\
&= \frac{2Qh^2}{\sqrt{\pi x K_{z,h} \overline{u}_h}}.
\end{aligned}$$

Then, the mean plume height:

$$\begin{aligned}
\overline{Z} &= \frac{\int_0^\infty \bar{c} z\, dz}{\int_0^\infty \bar{c}\, dz} \\
&= 2\sqrt{\frac{xK_{z,h}}{\pi \overline{u}_h}}.
\end{aligned}$$

The values of $\overline{Z}$ for different downwind distances using wind speed $= 2.3\ \mathrm{m\,s^{-1}}$ and eddy diffusivity $= 0.24\ \mathrm{m^2\,s^{-1}}$ are given in Figure 7.6.

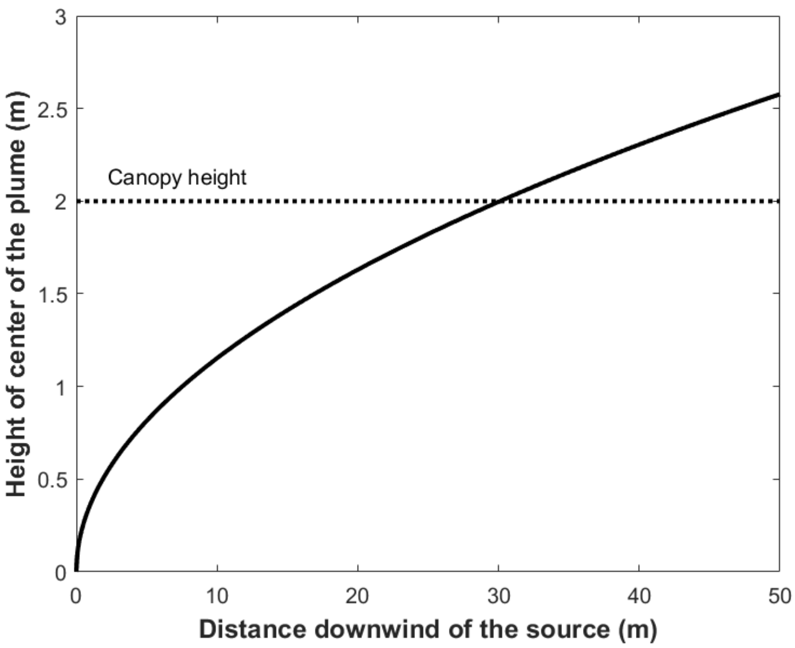


Figure 7.6: Mean plume height at downwind distances from the emission source

According to Figure 7.6, the mean plume height crosses the canopy height ($\overline{Z} = h = 2.0$ m) at 30.1 m downwind of the emission source.

7.14 Evaluate the one-dimensional footprint function Equation 7.71 for measurement heights of 3.0 m and 9.0 m, with surface roughness of 0.065 m. Present your result in a graphic plot. How does measurement height affect the flux footprint? Repeat the calculation with $z_o = 0.3$ m. How does surface roughness affect the flux footprint?

When $z_0 = 0.065$ m: $z_u = 8.56$ m for $z_m = 3.0$ m and $z_u = 35.44$ m for $z_m = 9.0$ m.

When $z_0 = 0.3$ m: $z_u = 4.21$ m for $z_m = 3.0$ m and $z_u = 21.91$ m for $z_m = 9.0$ m.

The footprint functions according to Original Equation 7.71 are shown in Figure 7.7. As seen in Figures 7.7, increasing z_m will increase the contribution from farther surface sources while decrease the contribution from closer surface sources; meanwhile, the maximum contribution source is shifted farther away from the sensor. Comparing the these plots indicates that a larger roughness corresponds to a narrower footprint function, that is, more contributions from sources closer to the sensor.

7.15 Original Equation 7.32 is the solution for a ground-level line source in homogeneous turbulence. Derive the footprint function using this equation and propose a method for calculating the mean eddy diffusivity and the mean plume velocity in the surface layer. Compare your footprint model with the model described by Original Equation 7.71 for surface roughness of 0.04 m, a measurement height of 4.0 m and neutral stability.

The first step is to get the concentration resulting from a line source of unit strength $\overline{c}_1$ from Original Equation 7.32:

$$\overline{c}_1 = \frac{\overline{c}(x,z)}{Q} = \frac{\exp\left(-\frac{u_p z^2}{4K_z x}\right)}{(\pi K_z u_p x)^{\frac{1}{2}}}$$

According to Original Equation 7.65, the one-dimensional footprint function is given by:

$$f_1(x; z_m) = -K_z \frac{\partial \overline{c}_1}{\partial z}\bigg|_{z_m} = \frac{\frac{u_p z_m}{2x}}{(\pi K_z u_p x)^{\frac{1}{2}}} \exp\left(-\frac{u_p z_m^2}{4K_z x}\right)$$

In the surface layer and under neutral condition, the eddy diffusivity is given by:

$$K_z = k u_* z$$

We propose that the the mean value of diffusivity be used for this footprint function, as

$$\overline{K}_z = \frac{\int_{z_0}^{z_m} k u_* z dz}{z_m - z_0} = \frac{z_m + z_0}{2} k u_*$$

The mean plume velocity is approximated by the mean wind speed below the sensor height (Original Equation 7.69)

$$u_p = \frac{u_* z_u}{k z_m}$$

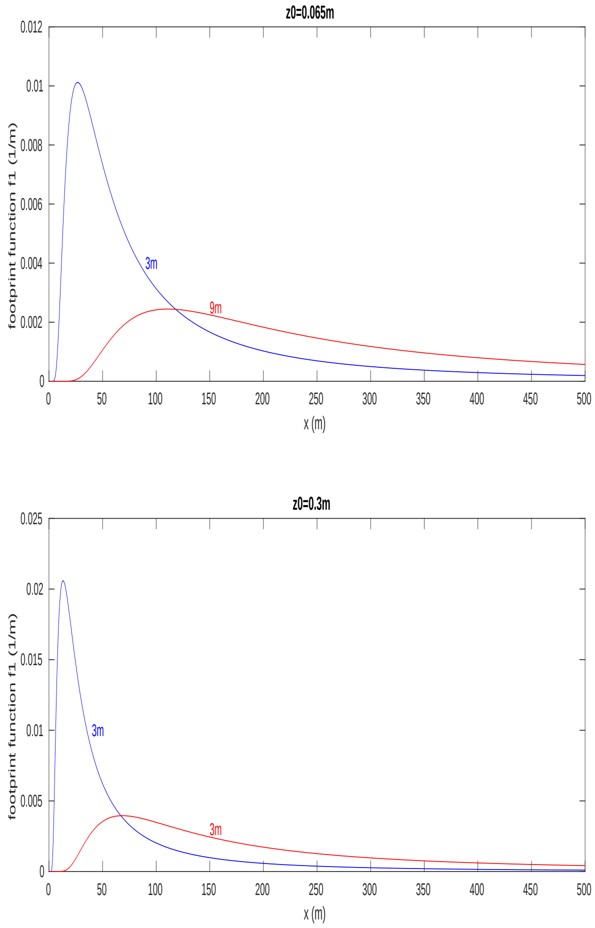


Figure 7.7: Footprint functions for two roughness values: red – measurement height = 9 m; blue – measurement height = 3 m.

Therefore, the footprint function derived from Original Equation 7.32 is finally:

$$f_1(x; z_m) = \frac{\frac{z_u}{2kx}}{\left(\frac{\pi(z_0+z_m)z_u x}{2z_m}\right)^{\frac{1}{2}}} \exp\left(-\frac{z_u z_m}{2(z_0+z_m)k^2 x}\right) \tag{7.8}$$

Given surface roughness of 0.04 m and the measure height of 4 m, z_u is approximately 14.46 m. The above footprint function is:

$$f_1(x; z_m) = \frac{3.77}{x^{\frac{3}{2}}} \exp\left(-\frac{44.74}{x}\right)$$

For comparison, Original Equation 7.71 is simplified to

$$f_1(x; z_m) = \frac{90.375}{x^2} \exp\left(-\frac{90.375}{x}\right)$$

A comparison of these two footprint functions is shown in Figure 7.8.

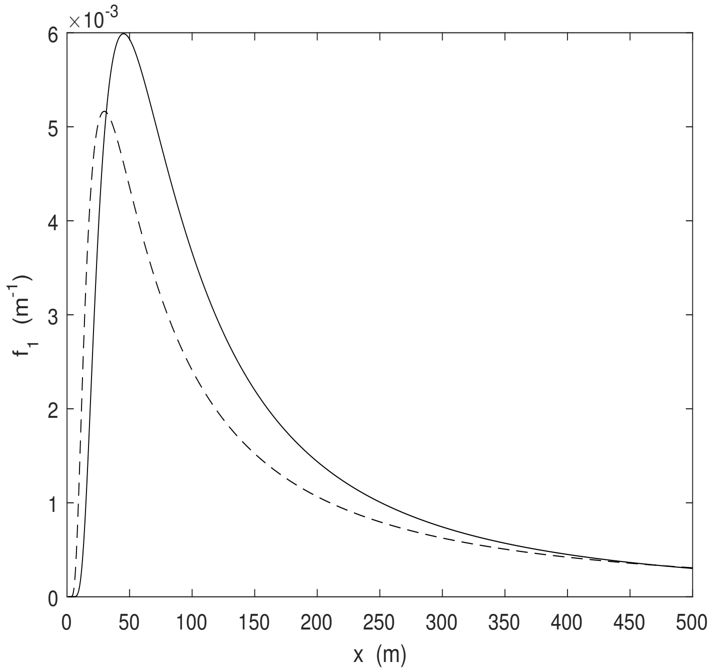


Figure 7.8: Comparison of footprint model Equation 7.8 (dashed line) with the original footprint model for neutral stability (Original Equation 7.71, solid line)

7.16 Show that the footprint model consisting of Equations 7.43, 7.45 and 7.75 is independent of the surface friction velocity.

Original Equation 7.43 does not involve friction velocity u_*:

$$\frac{\mathrm{d}\overline{Z}}{\mathrm{d}x} = \frac{k^2}{[\ln\bigl(p\overline{Z}/z_0\bigr) - \Psi_h(p\overline{Z}/L)]\phi_h(p\overline{Z}/L)}$$

Original Equation 7.45 indicates that the mean plume velocity is always proportional to friction velocity u_*:

$$u_p = \begin{cases} \frac{u_*}{k}[\ln\bigl(0.6\overline{Z}/z_0\bigr) + 4.7\overline{Z}/L], & \text{if } \zeta > 0 \\ \frac{u_*}{k}\ln\bigl(0.6\overline{Z}/z_0\bigr), & \text{if } \zeta = 0 \\ \frac{u_*}{k}[\ln\bigl(0.6\overline{Z}/z_0\bigr) - \Psi_h(0.6\overline{Z}/L], & \text{if } \zeta < 0 \end{cases}$$

In Original Equation 7.75, the friction velocity term u_* in the denominator cancels that in mean plume velocity in the numerator. So the footprint function is independent of friction velocity.

7.17 Verify that Equations 7.71 and 7.74 satisfy the integral constraint Equation 7.59.

Inserting Original Equation 7.71 into Original Equation 7.59, we obtain

$$\int_0^{+\infty} \frac{z_u}{k^2x^2}\exp\Bigl(-\frac{z_u}{k^2x}\Bigr)dx = \exp\Bigl(-\frac{z_u}{k^2x}\Bigr)|_0^{+\infty} = 1 - 0 = 1$$

Similarly, inserting Original Equation 7.74 into Original Equation 7.59, we have

$$\int_0^{+\infty} \frac{Dz_u^b|L|^{1-b}}{k^2x^2} \exp\Big(-\frac{Dz_u^b|L|^{1-b}}{k^2x}\Big) = \exp\Big(-\frac{Dz_u^b|L|^{1-b}}{k^2x}\Big)\Big|_0^{+\infty} = 1 - 0 = 1$$

Therefore, these two equations satisfy the integral constraint.

7.18 Evaluate the one-dimensional footprint function Equation 7.74 for three stability classes (neutral stability, L = 100 m, and $L = -50$ m) for a measurement height of 4.0 m and surface roughness of 0.04 m. Present your result in a graphic plot. How does air stability affect the flux footprint?

We use Original Equation 7.74:

$$f_1(x; z_m) = \frac{Dz_u^b|L|^{1-b}}{k^2x^2} \exp\Big(-\frac{Dz_u^b|L|^{1-b}}{k^2x}\Big)$$

We are given: $z_m = 4.0$ m, and $z_o = 0.04$ m. We calculate z_u:

$$z_u = z_m[\ln(z_m/z_o) - 1 + z_o/z_m] = 4.0[\ln(4.0/0.04) - 1 + 0.04/4.0] = 14.46 \text{ m}$$

For neutral stability, Original Equation 7.74 is reduced to Original Equation 7.71:

$$f_1(x; z_m) = \frac{z_u}{k^2x^2} \exp\Big(-\frac{z_u}{k^2x}\Big)$$

For L = 100 m (stable), D = 2.44 and b = 1.33. For $L = -50$ m (unstable), D = 0.28 and b = 0.59. These functions are shown graphically in Figure 7.9.

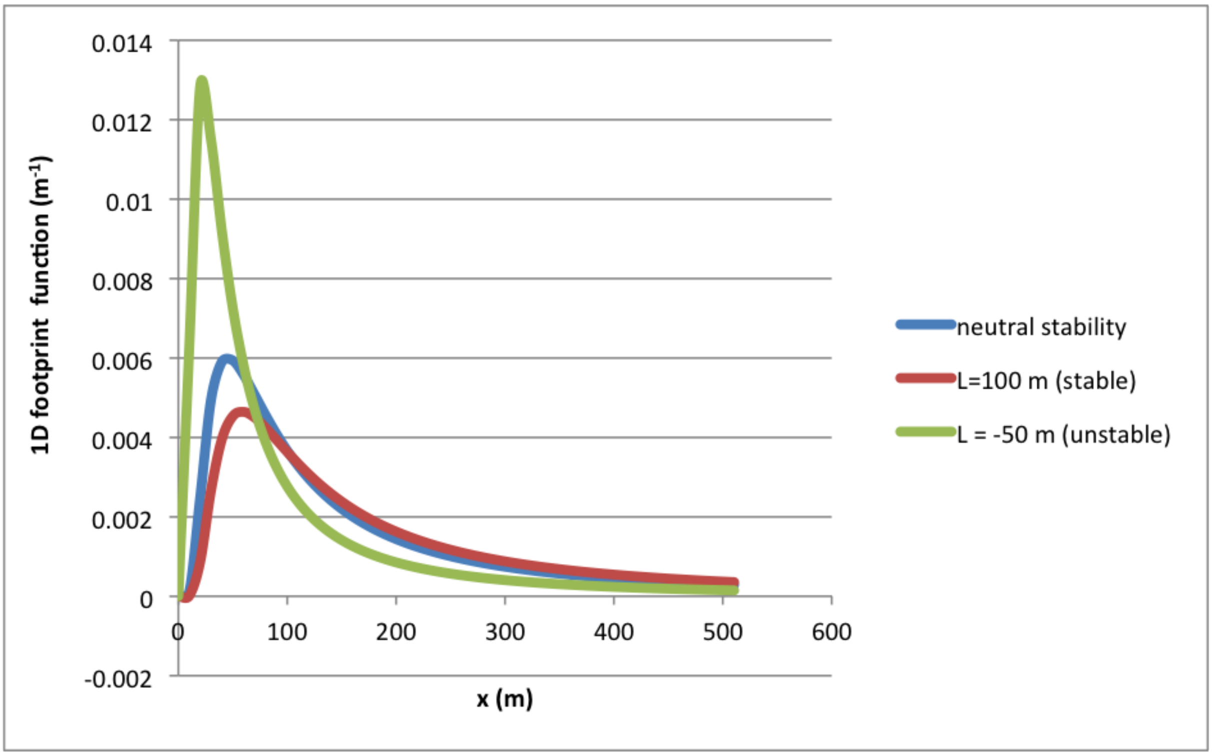


Figure 7.9: One dimensional footprint function under three stability classes (green, unstable; blue, neutral; red, stable)

7.19* Using a numerical procedure, check the footprint function Equation 7.75 against the integral constraint Equation 7.59 for a measurement height of 4 m, surface roughness of 0.04 m and neutral stability. Explain why the integral of Equation 7.75 with respect to distance

$$\int_0^x f_1(x', z_m)\mathrm{d}x'$$

does not seem to converge to unity as x increases.

According to Original Equations 7.75 and 7.45 and $\phi_h(z_m/L) = 1$ in neutral condition, we have:

$$f_1(x, z_m) = \frac{ku_* Ar}{\overline{Z}u_p}\left(\frac{Bz_m}{\overline{Z}}\right)^r \exp\left[-\left(\frac{Bz_m}{\overline{Z}}\right)^r\right]$$

where

$$u_p = \frac{u_*}{k}\ln\left(0.6\overline{Z}/z_0\right)$$

We are given: $r = 0.5$, $A = 0.73$, $B = 0.66$, $z_m = 4$ m, and $z_0 = 0.04$ m. The mean plume height $\overline{Z}$ is found for a given x by inverting Original Equation 7.44.

The integral $\int_0^x f_1 \mathrm{d}x'$ is evaluated numerically in Matalb and is shown in Figure 7.10 as a function of x. As x increase, this integral does not converge to unity.

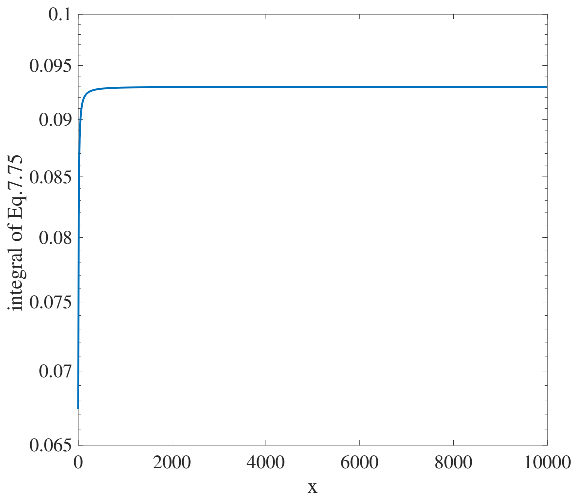


Figure 7.10: Integration of Equation 7.75 as a function of x (in m)

7.20* You want to measure the evapotranspiration flux of a hayfield with an eddy-covariance system. The hay field has a fetch, or distance between the instrument tower and the upwind edge of the field, of 160 m. You plan to install the instrument at a height of 2.5 m above the surface. Your footprint threshold is 90%, meaning that at least 90% of the measured evapotranspiration flux should come from the hayfield. Using the footprint function Equation 7.71, show that your experimental plan does not satisfy the requirement. To what height should you lower the instrument to ensure that the requirement is met?

Let us assume that the hayfield has a vegetation height h of 0.5 m, a displacement height $d = 0.666h$ and a roughness length $z_0 = 0.123h$. The cumulative normalized flux can be calculated by integrating Original Equation 7.71. In this calculation, the height scale should adjusted for the displacement height:

$$z_u = (z_m - d)\big[\ln\left(\frac{z_m - d}{z_0}\right) - 1 + \frac{z_0}{z_m - d}\big]$$

The result is shown in Figure 7.11. If we install the instrument at a height of 2.5 m, only about 80% of the flux comes from the range of 0 to 160 m. To increase the contribution to 90%, we should install the instrument at a height of about 1.5 m above the ground.

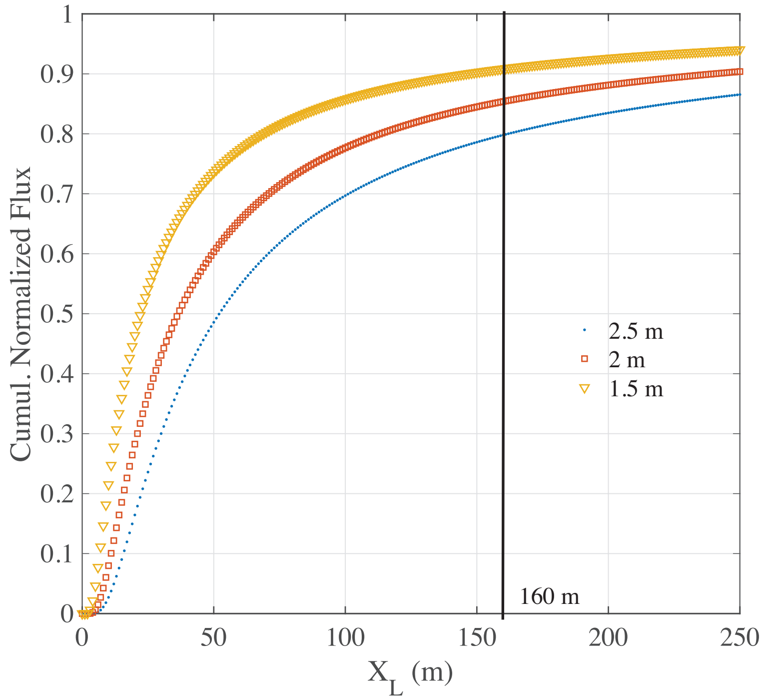


Figure 7.11: Cumulative flux contribution to the flux measured at three heights indicated

Chapter 8

Principle of Eddy Covariance

8.1 Show that the canopy CO_2 source term $\overline{S}_{c,p}$ has the dimensions of $\mathrm{kg\,m^{-3}s^{-1}}$.

The canopy CO_2 source term is:

$$\overline{S}_{c,p} = -2\overline{\rho}_d \kappa_c a \left[\frac{\partial \overline{s}_c}{\partial n}\right]$$

where $\overline{\rho}_d$ has dimension of kg $\mathrm{m^{-3}}$, κ_c has dimension of $\mathrm{m^2}$ $\mathrm{s^{-1}}$, a has dimension of $\mathrm{m^{-1}}$, and $\frac{\partial \overline{s}_c}{\partial n}$ has dimension of $\mathrm{m^{-1}}$. The dimensions of the product of these terms simplify to kg $\mathrm{m^{-3}s^{-1}}$.

8.2 Calculate the net ecosystem sensible heat exchange using Equations 8.10 and 8.14 under these conditions: the leaf temperature is 23.0°C, the leaf boundary layer thickness is 2 mm, the air temperature outside the leaf boundary layer is 21.0°C, the mean leaf area density is 0.20 $\mathrm{m^2m^{-3}}$, the canopy height is 20.0 m, and the soil sensible heat source is negligible.

According to Original Equation 8.10, the canopy sensible heat source term is given by:

$$\overline{S}_{T,p} = -2\kappa_T a[\frac{\partial \overline{T}}{\partial n}] \tag{8.1}$$

According to Original Equation 8.14, net ecosystem exchange of heat is given by:

$$H = \int_0^h \overline{\rho}_d c_p \overline{S}_{T,p} dz' \tag{8.2}$$

We can combine these two equations to solve for the net ecosystem exchange of heat:

$$H = \int_0^h \overline{\rho}_d c_p(-2\kappa_T a[\frac{\partial \overline{T}}{\partial n}])dz'$$

$$= h\overline{\rho}_d c_p(-2\kappa_T a[\frac{\partial \overline{T}}{\partial n}])$$

$$= (20\mathrm{m})(1.225\mathrm{kgm^{-3}})(1.00\mathrm{kJkg^{-1}K^{-1}})(-2)(1.9\times 10^{-5}\mathrm{m^2s^{-1}})(0.20\mathrm{m^2m^{-3}})(\frac{(21.0-23.0)\mathrm{K}}{0.002\mathrm{m}})$$

$$= 186 \text{ Wm}^{-2}$$

8.3 Consider a canopy water vapor source strength $\overline{S}_{v,p}$ profile given by,

$$\overline{S}_{v,p} = \frac{0.1}{\sqrt{2\pi} \cdot 3} \exp\left[-\frac{(z-15)^2}{2 \cdot 3^2}\right], \tag{8.3}$$

where $\overline{S}_{v,p}$ is in $\mathrm{g\,m^{-3}\,s^{-1}}$, and z is height above the ground in m. What is the net ecosystem water vapor exchange? What is the corresponding latent heat flux?

According to Original Equation 8.13, net ecosystem water vapor exchange is given by:

$$E = \int_0^h \overline{S}_{v,p} dz'$$

Substituting the canopy water vapor source strength profile (Equation 8.3) yields:

$$E = \int_0^h \frac{0.1}{\sqrt{2\pi} \times 3} \exp\left[\frac{(z-15)^2}{2 \times 3^2}\right] dz' = 0.1 \int_0^h \frac{1}{\sqrt{2\pi} \times 3} \exp\left[\frac{(z-15)^2}{2 \times 3^2}\right] dz'$$

We note that the function being integrated is a normal distribution function whose integral from 0 to $+\infty$ is 1/2. So we find that E is approximately 0.05 g $\mathrm{m^{-2}}$ $\mathrm{s^{-1}}$.

Latent heat flux is given by the relationship:

$$LE = \lambda E \tag{8.4}$$

So, multiplying E with latent heat of vaporization (2466 $\mathrm{J\,g^{-1}}$) gives latent heat flux of 123.3 $\mathrm{W\,m^{-2}}$

8.4 A closed chamber of height 24 cm is used to measure the CO_2 flux of a forest soil. The CO_2 molar mixing ratio in the chamber is 402.1 ppm right after the chamber is placed on the soil, and increases to 433.6 ppm 60 s later. The air temperature is 11.5°C. What is the CO_2 flux?

Net ecosystem exchange of CO_2 is given by Original Equation 8.16:

$$\text{NEE} = \overline{\rho}_d d \frac{\partial \overline{s}_{c,c}}{\partial t}$$

To use this equation, we need to convert the molar mixing ratios to mass mixing ratios using Original Equations 2.34 and 2.36. The initial mass mixing ratio is:

$$402.1 \times \frac{44}{29} = 610.08 \times 10^{-6}$$

The mass mixing ratio at 60 s later is:

$$433.6 \times \frac{44}{29} = 657.88 \times 10^{-6}$$

So we have

$$\text{NEE} = (1.225\,\text{kg}\,\text{m}^{-3})(0.24\text{ m})\frac{657.88 - 610.08}{60\text{ s} - 0\text{ s}} \times 10^{-6} = 0.235\text{ mg}\,\text{m}^{-2}\,\text{s}^{-1}$$

8.5 An opaque Teflon box of dimensions 30 cm by 60 cm by 30 cm (width by length by height) is used to measure mercury emission of a forest soil. The bottom face of the box is removed. The box is placed on the soil surface. Air is drawn into the box through holes cut into the front at a base flow rate of 11.5 $\text{L}\,\text{min}^{-1}$. The ambient gaseous mercury concentration is 1.80 $\text{ng}\,\text{m}^{-3}$ at STP (standard temperature and pressure) and the concentration of the chamber outlet air is 2.53 $\text{ng}\,\text{m}^{-3}$ at STP. Determine the gaseous mercury flux. Express your result in $\text{ng}\,\text{m}^{-2}\,\text{h}^{-1}$.

We can use Original Equation 8.19 to find the gaseous mercury flux:

$$F_m = \overline{\rho}_d \frac{Q_b}{A_b}(\overline{s}_{m,o} - \overline{s}_{m,i}) \tag{8.5}$$

Because this equation requires mass mixing ratio, we should convert the mercury mass density ρ_m to its mass mixing ratio s_m, as:

$$s_m = \rho_m / \rho_d$$

Combining the above two equations, we have

$$F_m = \frac{Q_b}{A_b}(\overline{\rho}_{m,o} - \overline{\rho}_{m,i})$$

$$= \frac{(0.0115 \times 60)\,\text{m}^3\text{h}^{-1}}{(0.30 \times 0.60)\,\text{m}^2}(2.53\,\text{ng}\,\text{m}^{-3} - 1.80\ \ \text{ng}\,\text{m}^{-3}) = 2.80\ \ \text{ng}\,\text{m}^{-2}\,\text{h}^{-1}$$

In the above calculation, we have not considered the dilution or density effect due to soil evaporation on the measured mercury mass density. (There is no density effect due to temperature variations because the measurement is measured at the standard temperature.)

8.6 You want to determine the optimal flow rate for a dynamic flux chamber for measurement of the net ecosystem methane exchange of a rice paddy ecosystem. Your chamber has a basal area of 0.1 m^2 and your analyzer has a molar mixing ratio precision of 2 ppb. The expected net exchange rate is 2 $\mu\text{g}\,\text{CH}_4\,\text{m}^{-2}\text{s}^{-1}$ according to the published literature. At a base flow rate of 50 $\text{L}\,\text{min}^{-1}$, can your analyzer resolve the concentration difference between the chamber outlet and the inlet? How should the flow rate be adjusted to reduce measurement error?

We use Original Equation 8.19 to solve this problem:

$$\text{NEE} = \overline{\rho}_d \frac{Q_b}{A_b}(\overline{s}_{c,o} - \overline{s}_{c,i})$$

which we can solve for $\overline{s}_{c,o} - \overline{s}_{c,i}$ We know: $\overline{\rho}_d = 1.2041$ kg m^{-3}, $Q_b = 50$ L min^{-1} = 0.00083 m^3 s^{-1}, $A_b = 0.1$ m^2, NEE = 2 μg CH_4 m^{-2}s^{-1}. Equation 8.19 becomes:

$$2 \text{ μg m}^{-2}\text{ s}^{-1} = 1.2041 \text{ kg m}^{-3} \times \frac{0.00083}{0.1} \text{ m s}^{-1} \times (\overline{s}_{c,o} - \overline{s}_{c,i}).$$

We obtain:

$$(\overline{s}_{c,o} - \overline{s}_{c,i}) = 2.00 \times 10^{-7} \text{ kg kg}^{-1}$$

Converting mass mixing ratio to molar mixing ratio (Original Equations 2.34 and 2.36), we obtain the molar mixing ratio difference between the chamber outlet and inlet:

$$2.00 \times 10^{-7} \times \frac{29}{16} = 3.63 \times 10^{-7} = 363 \text{ ppb} >> 2 \text{ ppb}$$

(Note that we have used the molar mass of 16 g for methane and 29 g for dry air.) So the gas analyzer's precision is good enough for this measurement.

A slower flow rate would result in a larger difference between the inlet and the outlet while a faster flow rate would result in a smaller difference. Slower flow rates may reduce measurement error because the concentration difference is much larger than the instrument precision.

8.7 The nitrous oxide emission rate of a typical fertilized corn soil in the U. S. Midwest is 0.3 nmol m^{-2} s^{-1}. You want to measure the emission with a dynamic chamber that has a basal area of 0.25 m^2 and a base flow rate of 10 L min^{-1}. How large is the expected concentration difference (in ppb) between the chamber outlet and the inlet? If you attempt to measure the emission with the flux-gradient method, how large do you expect the vertical concentration difference to be between the heights of 2.5 and 3.5 m above the ground? Assume that the canopy height is 2.0 m and the surface friction velocity is 0.25 m s^{-1}. Which of the two methods requires much better instrument precision?

We use Original Equation 8.19 to solve this problem:

$$\text{NEE} = \overline{\rho}_d \frac{Q_b}{A_b} (\overline{s}_{c,o} - \overline{s}_{c,i})$$

We can solve for $\overline{s}_{c,o} - \overline{s}_{c,i}$, using the following givens: $\overline{\rho}_d = 1.2041$ kg m^{-3}, $Q_b = 10$ L min^{-1} = 0.00017 m^3 s^{-1}, $A_b = 0.25$ m^2, NEE = 0.3 nmol m^{-2} s^{-1} = 1.32 $\times 10^{-8}$ g m^{-2} s^{-1}, using the molar mass 44 g mol^{-1}. We obtain:

$$\overline{s}_{c,o} - \overline{s}_{c,i} = 16.1 \times 10^{-9} \text{ g g}^{-1}$$

Converting mass mixing ratio to molar mixing ratio (Original Equations 2.34 and 2.36), we obtain the molar mixing ratio difference between the chamber outlet and inlet:

$$16.1 \times 10^{-9} \times \frac{29}{44} = 10.5 \text{ ppb}$$

We use an equation similar to Original Equation 3.64 to calculate the vertical gradient:

$$F_{n2o} = -\overline{\rho}_d K_c \frac{\overline{s}_{c,2} - \overline{s}_{c,1}}{z_2 - z_1}$$

We calculate:

$$z_g = [(z_2 - d)(z_1 - d)]^{1/2}$$

where $d \simeq 2.0$ m. We calculate:

$$K_c = \frac{k z_g u_*}{\phi_h}$$

We will assume neutral stability ($\phi_h = 1$), so $K_c = 0.087\ \mathrm{m^2\,s^{-1}}$. Filling in these values, we obtain:

$$\overline{s}_{c,2} - \overline{s}_{c,1} = 0.1266 \times 10^{-9}\ \mathrm{g\,g^{-1}}$$

Converting to molar mixing ratio, we estimate that the molar mixing ratio difference between the lower and upper measurement height is:

$$0.1266 \times 10^{-9} \times \frac{29}{44} = 0.083\ \text{ppb}$$

In this case, the flux-gradient approach requires much better instrument precision.

8.8 The column mean CO_2 molar mixing ratio between the ground and the eddy covariance sensor height of 30 m is 380.1 ppm at 18:00 and 395.2 ppm at 24:00 on September 1, and 398.4 ppm at 06:00, 385.0 ppm at 12:00 and 380.2 ppm at 18:00 on September 2. Calculate the CO_2 storage term for periods between 18:00 and 24:00 on September 1 and between 18:00, September 1 and 18:00, September 2. Express your results in $\mu\mathrm{mol\,m^{-2}\,s^{-1}}$.

Convert the molar mixing ratio to mass mixing ratio using Original Equation 2.34:

$$s_c = \frac{M_c}{M_d}\chi_c = \frac{0.044\ \text{kg mol}^{-1}}{0.029\ \text{kg mol}^{-1}}(\chi_c)$$

September 1 at 18:00, $s_c = 5.767 \times 10^{-4}$ and at 24:00, $s_c = 6.00 \times 10^{-4}$; September 2 $s_c = 6.04 \times 10^{-4}$ at 06:00, $s_c = 5.84 \times 10^{-4}$ at 12:00, and $s_c = 5.769 \times 10^{-4}$ at 18:00

Using Term I of Original Equation 8.20, CO_2 storage term between 18:00 and 24:00 on September 1:

$$\begin{aligned}
&= \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_c}{\partial t} dz' \\
&= (1.2041\ \mathrm{kg\,m^{-3}}) \times \frac{6.00 \times 10^{-4} - 5.767 \times 10^{-4}}{(24 - 18\ \text{hr})(3600\ \text{s})} \times (30\ \text{m} - 0\ \text{m}) \\
&= 3.85 \times 10^{-5}\ \text{g}\ \mathrm{m^{-2}\,s^{-1}} \times \frac{1\ \text{mol}}{44\ \text{g}} \\
&= 0.874\ \mu\text{mol m}^{-2}\ \text{s}^{-1}
\end{aligned}$$

Similarly, CO_2 storage term between 18:00 on September 1 and 18:00 on September 2:

$$\begin{aligned}
&= (1.2041 \text{ kg m}^{-3}) \times \frac{5.769 \times 10^{-4} - 5.767 \times 10^{-4}}{(24 \text{ hr})(3600 \text{ s})} \times (30 \text{ m} - 0 \text{ m}) \\
&= 6.34 \times 10^{-8} \text{ g m}^{-2}\text{ s}^{-1} \times \frac{1 \text{ mol}}{44 \text{ g}} \\
&= 1.44 \times 10^{-3} \mu\text{mol m}^{-2} \text{ s}^{-1}
\end{aligned}$$

8.9 The mean air temperature and the water vapor mixing ratio in the air column below an eddy covariance system are 20.1°C and 17.2 mmol mol^{-1} at 08:00 and increase to 20.6°C and 17.4 mmol mol^{-1} at 09:00. Calculate the sensible and latent heat storage term in W m^{-2} for this time period. The eddy covariance measurement height is 2.4 m.

Use Term I of Original Equation 8.28 to get sensible heat storage term for 1 hour (3600 s):

$$\begin{aligned}
&= \int_0^z \overline{\rho}_d c_p \frac{\partial \overline{T}}{\partial t} dz' \\
&= (1.2041 \text{ kg m}^{-3}) \times (1004 \text{ J kg}^{-1} \text{ K}^{-1}) \times \frac{(293.75 - 293.25) \text{ K}}{3600 \text{ s}} \times (2.4 - 0) \text{ m} \\
&= 0.40 \text{ W m}^{-2}
\end{aligned}$$

For latent heat, we first convert the molar mixing ratio to mass mixing ratio using Original Equation 2.35:

$$\begin{aligned}
s_{v,1} &= \frac{M_v}{M_d}\chi_v = \frac{0.018 \text{ kg mol}^{-1}}{0.029 \text{ kg mol}^{-1}}(17.2 \text{ mmol mol}^{-1}) = 10.68 \text{ mg g}^{-1} \\
s_{v2} &= \frac{0.018 \text{ kg mol}^{-1}}{0.029 \text{ kg mol}^{-1}}(17.4 \text{ mmol mol}^{-1}) = 10.80 \text{ mg g}^{-1}
\end{aligned}$$

Next, we use Original Equation 8.29 to calculate the latent heat storage term:

$$\begin{aligned}
&= \lambda \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_v}{\partial t} dz' \\
&= (2466 \text{ J g}^{-1}) \times (1.2041 \text{ kg m}^{-3}) \times \frac{(10.80 - 10.68) \text{ mg g}^{-1}}{3600 \text{ s}} \times (2.4 - 0) \text{ m} \\
&= 0.24 \text{ W m}^{-2}
\end{aligned}$$

8.10 The eddy covariance terms are $\overline{w'\chi_c'} = -0.732$ ppm m s^{-1}, $\overline{w'\chi_v'} = 0.203$ mmol mol^{-1} m s^{-1}, and $\overline{w'T'} = 0.176$ K m s^{-1}. Assume that all other terms in the eddy covariance equations are negligible. What are the net ecosystem exchanges of CO_2 (in μmol m^{-2} s^{-1}), water vapor (in mmol m^{-2} s^{-1}) and sensible heat (in W m^{-2})?

We use Original Equation 8.24 to get net ecosystem exchange of CO_2, noting that the first term is negligible:

$$\begin{aligned}
\text{NEE} &= \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_c}{\partial t} dz' + \overline{\rho}_d \overline{w' s_c'} \\
&= \overline{\rho}_d \overline{w' s_c'} \\
&= (1.2041 \text{ kg m}^{-3}) \times (-0.732 \frac{\mu\text{mol}}{\text{mol}} \text{ m s}^{-1}) \frac{44 \text{ g mol}^{-1}}{29 \text{ g mol}^{-1}} \\
&= -1.34 \text{ mg m}^{-2}\text{ s}^{-1} \\
&= -1.34 \text{ mg m}^{-2}\text{ s}^{-1} \times \frac{1 \text{ mmol}}{44 \text{ mg}} = -30.4 \ \mu\text{mol m}^{-2}\text{ s}^{-1}
\end{aligned}$$

We use Original Equation 8.25 to get net ecosystem exchange of water vapor, again noting that the first term is negligible:

$$\begin{aligned}
E &= \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_v}{\partial t} dz' + \overline{\rho}_d \overline{w' s_v'} \\
&= \overline{\rho}_d \overline{w' s_v'} \\
&= (1.2041 \text{ kg m}^{-3}) \times (0.203 \text{ mmol mol}^{-1} \text{ m s}^{-1}) \frac{18 \text{ g mol}^{-1}}{29 \text{ g mol}^{-1}} \\
&= 0.152 \text{ g m}^{-2}\text{ s}^{-1} \\
&= 0.152 \text{ g m}^{-2}\text{ s}^{-1} \times \frac{1 \text{ mol}}{18 \text{ g}} = 8.43 \text{ mmol m}^{-2}\text{ s}^{-1}
\end{aligned}$$

We use Original Equation 8.26 to get net ecosystem exchange of sensible heat:

$$\begin{aligned}
H &= \int_0^z \overline{\rho}_d c_p \frac{\partial \overline{T}}{\partial t} dz' + \overline{\rho}_d c_p \overline{w' T'} \\
&= \overline{\rho}_d c_p \overline{w' T'} \\
&= (1.2041 \text{ kg m}^{-3}) \times (1004 \text{ J kg}^{-1}\text{K}^{-1}) \times (0.176 \text{ K m s}^{-1}) \\
&= 212.8 \text{ W m}^{-2}
\end{aligned}$$

8.11 Show that the pressure flux $\overline{w'p'}$ has the dimensions of W m^{-2}.

The dimension of the pressure flux is:

$$[\overline{w'p'}] = [\text{m s}^{-1}]\,[\text{N m}^{-2}] = [\text{J m}^{-2}\text{ s}^{-1}] = [\text{W m}^{-2}]$$

8.12 The plant area density is described by Original Equation 5.46, the plant area index is 3.0, the canopy drag coefficient is 0.2, the wind profile is given by Original Equation 5.27 ($\alpha_2 = 4.0$), and the wind speed at the top of the canopy is 1.5 m s^{-1}. Estimate the rate of heat generation by pressure compression in the canopy (Term IV, Original Equation 8.28).

The compression heat term is expressed as:

$$\int_0^z \overline{\rho}_d C_d a \overline{u}^3 dz' \tag{8.6}$$

The plant area density is:

$$a = \frac{L}{0.125h\sqrt{2\pi}} \exp\left(-\frac{(z/h - 0.65)^2}{2 \times 0.125^2}\right) = \frac{9.57}{h} \exp\left(-\frac{(z/h - 0.65)^2}{0.031}\right)$$

The wind profile is:

$$\overline{u}(z) = 1.5 \times \left[\frac{\sinh(4z/h)}{27.29}\right]^{\frac{1}{2}}$$

The air density $\overline{\rho}_d \simeq 1.20\ \mathrm{kg\,m^{-3}}$.

The integral (Equation 8.6) is evaluated by approximation in Matlab with summation from the ground to the top of the canopy:

$$\int_0^h \overline{\rho}_d C_d a \overline{u}^3 dz' = \overline{\rho}_d C_d \sum a_j \overline{u}_j^3 (\Delta z) = 0.38 \mathrm{W\,m^{-2}}$$

```
% --------------------
%Matlab code for problem 8.12
zh=[0:0.01:1];
rho_d=1.2;
C_d=0.2;
a=9.57.*exp(-(zh-0.65).^2/0.031);
u=1.5*(sinh(4.*zh)/27.29).^0.5;zh=[0:0.01:1];
heat=rho_d*C_d*sum(a.*u.^3)*0.01;
% --------------------
```

8.13 The landscape consists of two ecosystem types. Both are emitters of a tracer material to the atmosphere. The source strength of type I is N_1 and that of type II is N_2. According to footprint theory, the vertical flux of the tracer should lie somewhere between N_1 and N_2. However, the flux in the real world can fall outside the range bounded by N_1 and N_2. Why?

The flux footprint theory assumes that the flow is horizontally homogeneous. If vertical advection occurs, the measured flux can fall outside the range bounded by N_1 and N_2. For example, in the forest edge inflow scenario (Original Figure 8.8b), the measured CO_2 flux at night can be larger than the true NEE of either the open field or the forest (Problem 8.21).

8.14 Energy imbalance is defined as

$$\begin{aligned} I &= R_n - G - \left\{Q_s + \int_0^z \overline{\rho}_d c_p \frac{\partial \overline{T}}{\partial t} \mathrm{d}z' + \lambda \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_v}{\partial t} \mathrm{d}z'\right\} \\ &\quad - \left\{\overline{\rho}_d c_p \overline{w'T'} + \lambda \overline{\rho}_d \overline{w' s_v'}\right\}. \end{aligned} \tag{8.7}$$

Assume that the imbalance is caused by vertical advection. Determine if I is likely positive or negative for each of the flow types listed in Table 8.1.

To solve this problem, we rely on Original Equations 8.34 and 8.35, both of which include the vertical advection term:

$$E = \int_0^z \overline{\rho}_d \frac{\partial \overline{s}_v}{\partial t} dz' + \overline{\rho}_d \overline{w' s_v'} + \overline{\rho}_d \overline{w} (\overline{s}_v - < \overline{s}_v >)$$

and

$$H = \int_0^z \overline{\rho}_d c_p \frac{\partial \overline{T}}{\partial t} dz' + \overline{\rho}_d c_p \overline{w' T'} + \overline{\rho}_d c_p \overline{w} (\overline{T} - < \overline{T} >)$$

in both equations, the sign of the third terms on the right-hand side are the crucial step to judge the sign of I. If they are both positive, sensible heat and the latent heat fluxes are underestimated with the eddy covariance measurement without accounting vertical advection, and I is positive. Otherwise, I is negative.

Let us consider daytime convective conditions. In this case, both $\overline{T}$ and $\overline{s}_v$ show lower values when height increases, and $\overline{s}_v - < \overline{s}_v >$ and the $\overline{T} - < \overline{T} >$ are negative. Therefore, I is positive for synoptic subsidence and drainage flow, and is negative for sea/lake breeze and forest edge inflow. For stationary convection cells, I is positive in areas with downward air motion and is negative in areas with upward air motion.

8.15 In the presence of drainage flow, is the ecosystem respiration determined with the standard eddy covariance Equation 8.24 biased high or low? Provide an order-of-magnitude estimate of the bias error.

CO_2 tends to accumulate near the ground at night due to soil respiration, making $\partial \overline{s}_c / \partial z$ negative. The term $\overline{s}_c - < \overline{s}_c >$ is negative, and the NEE is underestimated in drainage flow ($\overline{w} < 0$).

Assuming that the term $\overline{s}_c - < \overline{s}_c >$ is on the order of $-100\ \mu g\, g^{-1}$, and $\overline{w}$ on the order of $-1 \times 10^{-2}\ m\, s^{-1}$, the bias term $\overline{\rho}_d \overline{w} (\overline{s}_c - < \overline{s}_c >)$ is about 1 $mg\, m^{-2}\, s^{-1}$.

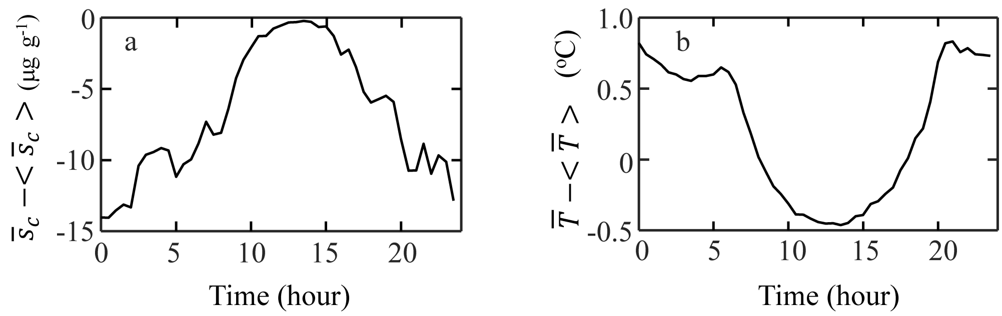


Figure 8.1: (Original Figure 8.12) Diurnal variations in the difference (a) between the CO_2 mixing ratio at the eddy covariance height ($\overline{s}_c$) and the column mean CO_2 mixing ratio below the height ($<\overline{s}_c>$), and (b) between the air temperature at the measurement height ($\overline{T}$) and the column mean temperature ($<\overline{T}>$) in a temperate forest during a growing season.

8.16 Assume that the mean vertical velocity at the eddy covariance measurement height is 0.01 $\mathrm{m\,s^{-1}}$ in the daytime and -0.01 $\mathrm{m\,s^{-1}}$ at night. Using the data given in Figure 8.1, estimate the contribution of vertical advection to the CO_2 budget of the eddy covariance control volume for 12:00 and 00:00. Is the daily mean NEE obtained from Equation 8.24 biased high or low?

We use Original Equations 8.31 and 8.33 to estimate the contribution of vertical advection to the CO_2 budget. From Original Equation 8.31:

$$\int_0^z \overline{\rho}_d \overline{w} \frac{\partial \overline{s}_c}{\partial z} \mathrm{d}z' = \overline{\rho}_d \overline{w} (\overline{s}_c - \langle \overline{s}_c \rangle)$$

The estimation for term $(\overline{s}_c - \langle \overline{s}_c \rangle)$, from Fig. 8.12, is -0.5 μg g^{-1} at 12:00, and -14 μg g^{-1} at 0:00. The value for dry air mass density, $\overline{\rho}_d$, is 1.225 kg m^{-3}. The mean vertical velocity at the eddy covariance measurement height is 0.01 $\mathrm{m\,s^{-1}}$ and -0.01 $\mathrm{m\,s^{-1}}$ at noon and at night, respectively. So the estimated contribution of vertical advection at noon is:

$$\overline{\rho}_d \overline{w} (\overline{s}_c - \langle \overline{s}_c \rangle) = 1.225 \times 10^3 \times 0.01 \times -0.5 = 6.125 \ \mu\mathrm{g\ m^{-2} s^{-1}}$$

The contribution is 171.5 μg m^{-2}s^{-1} at night.

The daily mean NEE obtained from Original Equation 8.24 is biased low (or towards a more negative value), since it does not consider the contribution of vertical advection, which is averaged to be positive during a day.

8.17 Repeat the calculations in Problem 8.16 but for sensible heat. How does the advection effect bias the daily mean ecosystem sensible heat exchange?

To estimate the contribution of vertical advection to the sensible heat budget, we use a modified equation from Original Equation 8.31, along with Original Equation 8.35:

$$\int_0^z \overline{\rho}_d c_p \overline{w} \frac{\partial \overline{T}}{\partial z} \mathrm{d}z' = \overline{\rho}_d c_p \overline{w} (\overline{T} - \langle \overline{T} \rangle)$$

where $(\overline{T} - \langle \overline{T} \rangle)$ is estimated to be -0.45 K at 12:00 and 0.80 K at 0:00. The value for dry air mass density, $\overline{\rho}_d$, is 1.225 kg m^{-3}. The mean vertical velocity at the eddy covariance measurement height is 0.01 $\mathrm{m\,s^{-1}}$ and -0.01 $\mathrm{m\,s^{-1}}$ at noon and at night, respectively. Repeating the calculation, we estimate that the contribution of vertical advection is -5.5 $\mathrm{W\,m^{-2}}$ at noon and is -9.8 $\mathrm{W\,m^{-2}}$ at night.

If the vertical advection term is ignored, as in Original Equation 8.26, the result will be biased high.

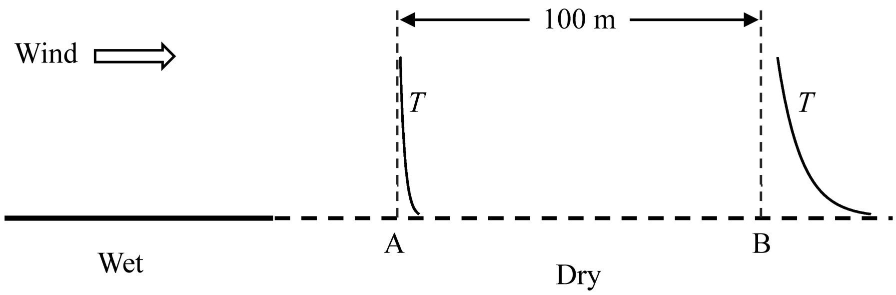


Figure 8.2: (Original Figure 8.13) Temperature profile at two locations in a dry field.

Table 8.1: (Original Table 8.2) Air temperature (T, °C) observed at position A and B shown in Figure 8.2. The separation distance between A and B is 100 m.

z (m)	0.1	0.5	1.0	1.5	2.0
A	20.31	20.18	20.16	20.09	20.07
B	21.53	20.90	20.63	20.47	20.36

8.18 (a) Write an equation similar to Original Equation 8.22 for horizontal heat advection. (b) Using the air temperature profile data given in Original Table 8.2, calculate the contribution of horizontal advection to the local heat budget at position B shown in Original Figure 8.13. Assume that the wind profile is logarithmic with height, the friction velocity is 0.30 $\mathrm{m\,s^{-1}}$, the surface momentum roughness is 0.05 m, and the eddy covariance measurement height is 2.0 m.

Table 8.2: Numerical value of $f(z')$ at each height

z' (m)	0.1	0.5	1.0	1.5	2.0
$f(z')$ ($\mathrm{W\,m^{-3}}$)	-7.80	-15.29	-12.99	-11.9	-9.8

An equation equivalent to Original Equation 8.22 for horizontal heat advection is:

$$\frac{1}{L}\int_0^L (\mathrm{H})\,\mathrm{d}x' = \frac{1}{L}\int_0^L\int_0^z \overline{\rho}_d c_p \overline{u}\frac{\partial \overline{T}}{\partial x}\mathrm{d}z'\mathrm{d}x'$$
$$= \frac{1}{L}\left\{\int_0^z \overline{\rho}_d c_p \overline{u}\,\overline{T}\mathrm{d}z'\big|_{x=L} - \int_0^z \overline{\rho}_d c_p \overline{u}\,\overline{T}\mathrm{d}z'\big|_{x=0}\right\}$$

Since air temperature, $\overline{T}$, is dependent on height, we have to solve this problem using a numerical approach.

The value of the function to be integrated:

$$f(z') = \frac{1}{L}\left\{\overline{\rho}_d c_p \overline{u}\,\overline{T}\big|_{x=L} - \overline{\rho}_d c_p \overline{u}\,\overline{T}\big|_{x=0}\right\}$$

is found for each height using the corresponding value for $\overline{T}$ (from Original Table 8.2) and $\overline{u}$ (from Original Equation 3.47). The values for each height are in Table 8.2.

Then we use the chained trapezoidal rule for numerical integration, given by:

$$\int_{0.1}^{2} f(z')\mathrm{d}z' = \sum_{k=1}^{N}\frac{f(z'_{k-1}) + f(z'_k)}{2}(z'_{k-1} - z'_k)$$

Using this method, we get a horizontal advection contribution of -23.4 W $\mathrm{m^{-2}}$.

8.19 The contribution of horizontal advection to the CO_2 budget of the eddy flux control volume (Figure 8.3) is 4.5 $\mu\mathrm{mol\,m^{-2}\,s^{-1}}$. Using Equation 8.22, estimate the corresponding concentration difference between the upwind and downwind face of the control volume. The horizontal dimension L is 100 m, the eddy covariance height is 20 m, and the mean wind speed below the height is 2 $\mathrm{m\,s^{-1}}$. Can you measure the concentration difference with a broadband analyzer whose precision is typically no better than 0.2 ppm?

Let us assume that the wind speed does not vary with height below the eddy covariance tower. From Original Equation 8.22:

$$\frac{1}{L}\int_0^L (\mathrm{NEE})\,\mathrm{d}x' = \frac{1}{L}\int_0^L\int_0^z \overline{\rho}_d \overline{u}\frac{\partial \overline{s}_c}{\partial x}\mathrm{d}z'\mathrm{d}x'$$
$$= \frac{1}{L}\left\{\int_0^z \overline{\rho}_d \overline{u}\,\overline{s}_c\mathrm{d}z'\big|_{x=L} - \int_0^z \overline{\rho}_d \overline{u}\,\overline{s}_c\mathrm{d}z'\big|_{x=0}\right\}$$

After converting the advection flux from $\mu\mathrm{mol\,m^{-2}\,s^{-1}}$, to $\mathrm{mg\,m^{-2}\,s^{-1}}$, we have:

$$4.5\times 0.044 = \frac{1}{L}\left\{\overline{\rho}_d\overline{u}z(\overline{s}_c(L) - \overline{s}_c(0))\right\} = \frac{1}{100}\left\{1.225\times 10^3\times 2\times 20\times(\overline{s}_c(100) - \overline{s}_c(0))\right\}$$

We obtain:

$$\overline{s}_c(100) - \overline{s}_c(0) = 0.4 \times 10^{-3} \text{ mg g}^{-1} = 0.4 \times 10^{-3} \times \frac{29}{44} = 0.26 \text{ ppm}$$

The last step in the above calculation converts the mass mixing ratio to the molar mixing ratio.

The broadband analyzer can barely detect the concentration difference between the upwind and downwind faces of the control volume.

8.20 Find the effective fetch for the following conditions: neutral air stability, measurement height = 2.0 m, surface roughness = 0.0225 m.

The effective fetch is defined as the distance at which the fractional contribution a_H is no greater than a preset threshold. A typical threshold value is 0.1. According to the definition of fractional contribution for neutral stability:

$$a_H = 1 - \exp\left(-\frac{z_u}{k^2\, x}\right)$$

We need to calculate the height scale, z_u, as defined in Original Equation 7.70:

$$z_u = z_m \left[\ln(z_m/z_0) - 1 + z_0/z_m\right]$$

Inserting $z_m = 2.0\,\text{m}$ and $z_0 = 0.0225\,\text{m}$, we obtain a height scale of approximately 7.0 m, so the effective fetch, at the threshold value $a_h = 0.1$:

$$x = \frac{k^2}{z_u} \ln(1 - a_H) \simeq 415\,\text{m}$$

8.21 It is known that ecosystem respiration rarely exceeds 0.4 mg CO_2 m^{-2} s^{-1}. However, the eddy CO_2 flux measured over a tall forest at night can be as high as 1.0 mg m^{-2} s^{-1}. (The measurement tower is located near the boundary that separates the forest from a large hayfield.) Explain reason(s) for this anomaly.

If the wind is coming from the large hayfield towards the forest, the forest decelerates the air motion. This convergence upwind of the tower leads to a positive vertical advection (to maintain local mass balance), which contributes to the CO_2 flux measured by the tower. Thus, since the advection term is not corrected for in eddy covariance systems, the measured flux is biased high, in this case, making the magnitude biologically unreasonable.

8.22* Using a threshold of 0.1 for the fractional contribution of horizontal advection a_H and Original Equations 7.74 and 8.36, derive an expression for the depth of the internal boundary layer as a function of downwind distance from a step change in surface source strength (Original Figure 8.1). Graph your result for a range of surface roughness and stability values and discuss how surface roughness and air stability affect the development of the internal boundary layer.

Integration of Original Equation 7.74 with respect to x yields

$$a_H = 1 - \exp\left(-\frac{D z_u^b |L|^{1-b}}{k^2 x}\right) \tag{8.8}$$

where the height scale z_u is given by Original Equation 7.70:

$$z_u = z[\ln(z/z_o) - 1 + z_o/z_m]$$

With $a_H = 0.1$, Equation 8.8 expresses the depth of the internal boundary layer z as an implicit function of fetch x and can be evaluated graphically. Figure 8.3 presents the results for two surface roughness $z_o = 0.065$ m (smooth) and 0.3 m (rough) and three stability class (neutral, unstable with the Obukhov length $L = -50$ m, and stable with $L = 100$ m). The internal boundary layer grows much faster in unstable conditions than in neutral and stable conditions. The growth is faster for a rougher surface.

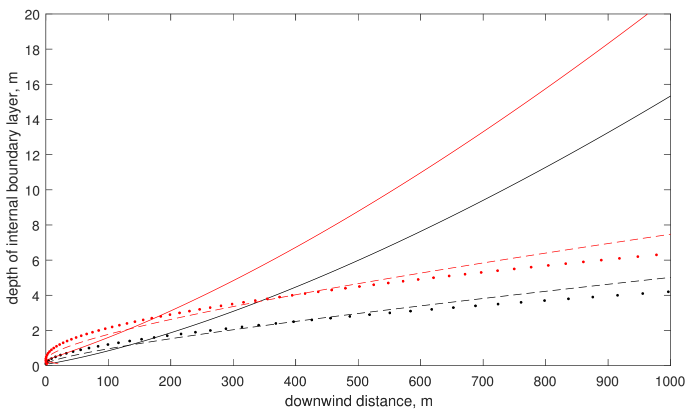


Figure 8.3: Depth of internal boundary layer as a function of fetch. Black lines are for a smooth surface and red lines are for a rough surface. solid lines: unstable; dashed lines: neutral; dotted lines: stable.

Chapter 9

Density Effects on Flux Measurements

9.1 Using the Taylor expansion and the Reynolds rules, show that (1) the Reynolds mean CO_2 mixing ratio can be approximated by $\overline{s}_c = \overline{\rho}_c/\overline{\rho}_d$, and (2) the mean dry air density can be approximated by Equation 9.14.

We start with Original Equation 9.6:

$$s_c = \frac{\overline{\rho}_c + \rho'_c}{\overline{\rho}_d + \rho'_d}$$

According to the Taylor expansion (Original Equation 9.8) and with omission of higher order terms, the denominator can be written as:

$$\frac{1}{\overline{\rho}_d + \rho'_d} = \frac{1}{\overline{\rho}_d}(1 - \frac{\rho'_d}{\overline{\rho}_d})$$

Combining the above two equations, we obtain:

$$s_c = \frac{\overline{\rho}_c}{\overline{\rho}_d}(1 + \frac{\rho'_c}{\overline{\rho}_c})(1 - \frac{\rho'_d}{\overline{\rho}_d})$$

Performing Reynolds averaging on the above equation, we get:

$$\begin{aligned}\overline{s}_c &= \frac{\overline{\rho}_c}{\overline{\rho}_d}(1 - \frac{\overline{\rho'_c\rho'_d}}{\overline{\rho}_c\overline{\rho}_d}) \\ &\simeq \frac{\overline{\rho}_c}{\overline{\rho}_d}\end{aligned}$$

For the second part, we start from Original Equation 9.13:

$$\rho_d = \frac{M_d(\overline{p} + p')}{R(\overline{T} + T')} - \mu(\overline{\rho}_v + \rho'_v)$$

Using Taylor expansion, the denominator of the first term becomes:

$$\frac{1}{R(\overline{T} + T')} = \frac{1}{R\overline{T}}(1 - \frac{T'}{\overline{T}})$$

Combining the two equations, we obtain:

$$\rho_d = M_d\overline{p}\frac{1}{R\overline{T}}(1-\frac{T'}{\overline{T}}) + M_d p'\frac{1}{R\overline{T}}(1-\frac{T'}{\overline{T}}) - \mu(\overline{\rho}_v + \rho'_v)$$

Performing Reynolds averaging on this equation, we get:

$$\begin{aligned}
\overline{\rho_d} &= \overline{M_d\overline{p}\frac{1}{R\overline{T}}(1-\frac{T'}{\overline{T}})} + \overline{M_d p'\frac{1}{R\overline{T}}(1-\frac{T'}{\overline{T}})} - \overline{\mu(\overline{\rho}_v + \rho'_v)} \\
&= \frac{M_d\overline{p}}{R\overline{T}} - \frac{M_d\overline{p'T'}}{R\overline{T}^2} - \mu\overline{\rho_v} \\
&\approx \frac{M_d\overline{p}}{R\overline{T}} - \mu\overline{\rho_v}
\end{aligned}$$

This is Original Equation 9.14

9.2 Find the dry air density for an air temperature of 15.3 °C, an air pressure of 952.3 hPa, and a water vapor density of 12.9 $\mathrm{g\,m^{-3}}$.

Approach 1:

Using the Dalton's law of partial pressures and the ideal gas law, we have:

$$\begin{aligned}
p &= p_v + p_d \\
p &= \rho_v R_v T + \rho_d R_d T \\
p &= \rho_v \frac{R}{M_v} T + \rho_d \frac{R}{M_d} T \\
\rho_d &= M_d(\frac{p}{RT} - \frac{\rho_v}{M_v}) \\
\rho_d &= 0.029 \times (\frac{952.3 \times 100}{8.314 \times 288.45} - \frac{12.9 \times 10^{-3}}{0.018}) \\
\rho_d &= 1.1308 \text{ kg m}^{-3}
\end{aligned}$$

Approach 2:

Alternatively, using Original Equation 9.12, the dry air density is:

$$\begin{aligned}
\rho_d &= \frac{M_d}{RT}p - \frac{M_d}{M_v}\rho_v \\
&= \frac{29 \times 952.3 \times 10^2}{8.314 \times (15.3 + 273.15)} - \frac{29}{18} \times 12.9 \\
&= 1.1308 \text{ kg m}^{-3}
\end{aligned}$$

Table 9.1: (Original Table 9.1) Instant air temperature (T, °C), water vapor density (ρ_v, g m^{-3}), CO_2 density (ρ_c, mg m^{-3}) and vertical velocity (w, m s^{-1}) measured with an open-path eddy covariance system at several time steps (t, s). The atmospheric pressure is 951.1 hPa.

t	1	2	3	4	5	6	7	8	9	10
T	20.47	20.47	20.44	20.56	20.52	20.50	20.87	21.01	20.72	20.79
ρ_v	7.367	7.401	7.403	7.359	7.313	7.343	7.375	7.369	7.425	7.424
$\rho_c - 600$	91.57	91.34	91.17	90.13	91.24	91.65	90.02	89.92	90.14	89.61
w	-0.010	0.730	0.375	0.612	0.998	1.919	1.945	1.991	2.155	1.106
t	11	12	13	14	15	16	17	18	19	20
T	20.68	20.57	20.43	20.10	20.08	20.09	20.10	20.10	20.13	20.39
ρ_v	7.346	7.370	7.195	7.138	7.109	7.086	7.077	7.144	7.244	7.285
$\rho_c - 600$	88.85	90.44	91.48	93.19	94.19	94.13	94.26	94.09	93.81	92.24
w	1.179	0.436	0.223	-0.217	-0.369	-0.494	-0.807	-0.890	-1.188	-0.988

9.3 The standard density correction procedure for eddy covariance is based on Reynolds mean statistics. The correction can also be made by first converting the instant mass density to the mass mixing ratio and then calculating the vertical velocity - mixing ratio covariance to obtain the true flux. Use both methods to compute the CO_2 flux with the short time series data shown in Table 9.1. Do the two methods produce nearly identical results? Why?

The mean dry air density is given by Original Equation 9.14:

$$\begin{aligned}\overline{\rho}_d &= \frac{M_d}{R\overline{T}}\overline{p} - \frac{M_d}{M_v}\overline{\rho}_v \\ &= \frac{29 \times 951.1 \times 10^2}{8.314 \times (293.601)} - \frac{29}{18} \times 7.29 \\ &= 1.118 \quad \text{kg m}^{-3}\end{aligned}$$

The mean CO_2 and water vapor mixing ratio are:

$$\overline{s}_c = \overline{\rho}_c/\overline{\rho}_d = 691.67 \times 10^{-6}/1.118 = 6.1855 \times 10^{-4}$$

$$\overline{s}_v = \overline{\rho}_v/\overline{\rho}_d = 7.29 \times 10^{-3}/1.118 = 6.5194 \times 10^{-3}$$

Using Original Equation 9.18:

$$\begin{aligned}\text{NEE} &= \overline{w'\rho_c'} + \overline{\rho}_c(1+\mu\overline{s}_v)\frac{\overline{w'T'}}{\overline{T}} + \mu\overline{s}_c(\overline{w'\rho_v'}) \\ &= -0.4125 \times 10^{-6} + 691.67 \times 10^{-6} \times (1 + 1.61 \times 6.5194 \times 10^{-3}) \times \frac{0.1005}{293.601} \\ &+ 1.61 \times 6.1855 \times 10^{-4} \times 0.02971 \times 10^{-3} \\ &= -1.4367 \times 10^{-7} \text{ kg m}^{-2}\text{ s}^{-1} = -0.1437 \text{ mg m}^{-2}\text{ s}^{-1}\end{aligned}$$

Another way to calculate the actual carbon dioxide flux is by calculating the $\overline{\rho_d}\overline{w's_c'}$. This is done by first calculating s_c using the instantaneous ρ_c values given in the table and the instantaneous ρ_d values found from Original Equation 9.12:

$$s_c = \frac{\rho_c}{\rho_d}$$

$$\rho_d = \frac{M_d\, p}{RT} - \mu \rho_v$$

Then, we find the perturbation, s_c', for each case by subtracting the mean of the instantaneous values from the individual instantaneous values. Finally, the covariance between w' and s_c' is found and its mean, multiplied by $\overline{\rho_d}$, gives the carbon dioxide flux. All calculations done for the problem are given in Tables 9.2 and 9.3.

Table 9.2

t	T	ρ_v	ρ_c	w	T'	ρ_v'	ρ_c'	w'
1	20.47	7.367	691.57	-0.01	0.019	0.07835	-0.1035	-1.149
2	20.47	7.401	691.34	0.73	0.019	0.11235	-0.3335	-0.409
3	20.44	7.403	691.17	0.375	-0.011	0.11435	-0.5035	-0.764
4	20.56	7.359	690.13	0.612	0.109	0.07035	-1.5435	-0.527
5	20.52	7.313	691.24	0.998	0.069	0.02435	-0.4335	-0.141
6	20.5	7.343	691.65	1.919	0.049	0.05435	-0.0235	0.78
7	20.87	7.375	690.02	1.945	0.419	0.08635	-1.6535	0.806
8	21.01	7.369	689.92	1.991	0.559	0.08035	-1.7535	0.852
9	20.72	7.425	690.14	2.155	0.269	0.13635	-1.5335	1.016
10	20.79	7.424	689.61	1.106	0.339	0.13535	-2.0635	-0.033
11	20.68	7.346	688.85	1.179	0.229	0.05735	-2.8235	0.04
12	20.57	7.37	690.44	0.436	0.119	0.08135	-1.2335	-0.703
13	20.43	7.195	691.48	0.223	-0.021	-0.09365	-0.1935	-0.916
14	20.1	7.138	693.19	0.217	-0.351	-0.15065	1.5165	-0.922
15	20.08	7.109	694.19	0.369	-0.371	-0.17965	2.5165	-0.77
16	20.09	7.086	694.13	0.494	-0.361	-0.20265	2.4565	-0.645
17	20.1	7.077	694.26	0.807	-0.351	-0.21165	2.5865	-0.332
18	20.1	7.144	694.09	0.890	-0.351	-0.14465	2.4165	-0.249
19	20.13	7.244	693.81	1.188	-0.321	-0.04465	2.1365	0.049
20	20.39	7.285	692.24	0.988	-0.061	-0.00365	0.5665	-0.151
Mean	20.451	7.28865	691.6735	1.139083	293.601			

Table 9.3

t	$w'\rho_c'$	$w'\rho_v'$	$w'T'$	ρ_d	ρ_d'	s_c	s_c'	$w's_c'$
1	0.118922	-0.09002	-0.02183	1.110357	-0.0002	0.000622836	1.94099E-08	-2.2302E-08
2	0.136401	-0.04595	-0.00777	1.110302	-0.00025	0.000622659	-1.57013E-07	6.42182E-08
3	0.384674	-0.08736	0.008404	1.110413	-0.00014	0.000622444	-3.72169E-07	2.84337E-07
4	0.813424	-0.03707	-0.05744	1.110028	-0.00053	0.000621723	-1.09344E-06	5.76244E-07
5	0.061123	-0.00343	-0.00973	1.110254	-0.0003	0.000622596	-2.20163E-07	3.1043E-08
6	-0.01833	0.042393	0.03822	1.110282	-0.00027	0.00062295	1.33653E-07	1.04249E-07
7	-1.33272	0.069598	0.337714	1.108828	-0.00173	0.000622297	-5.19441E-07	-4.18669E-07
8	-1.49398	0.068458	0.476268	1.108308	-0.00225	0.000622499	-3.17633E-07	-2.70624E-07
9	-1.55804	0.138532	0.273304	1.109315	-0.00124	0.000622132	-6.84808E-07	-6.95765E-07
10	0.068095	-0.00447	-0.01119	1.109052	-0.0015	0.000621801	-1.01482E-06	3.34891E-08
11	-0.11294	0.002294	0.00916	1.109594	-0.00096	0.000620813	-2.00376E-06	-8.01506E-08
12	0.86715	-0.05719	-0.08366	1.109973	-0.00058	0.000622033	-7.82979E-07	5.50434E-07
13	0.177246	0.085783	0.019236	1.110786	0.00023	0.000622514	-3.02149E-07	2.76769E-07
14	-1.39821	0.138899	0.323622	1.112132	0.001576	0.000623298	4.81842E-07	-4.44258E-07
15	-1.93771	0.138331	0.28567	1.112255	0.001699	0.000624128	1.31208E-06	-1.0103E-06
16	-1.58444	0.130709	0.232845	1.112254	0.001698	0.000624075	1.2587E-06	-8.11859E-07
17	-0.85872	0.070268	0.116532	1.112231	0.001674	0.000624205	1.3888E-06	-4.61081E-07
18	-0.60171	0.036018	0.087399	1.112123	0.001566	0.000624113	1.29652E-06	-3.22834E-07
19	0.104688	-0.00219	-0.01573	1.111847	0.001291	0.000624016	1.19921E-06	5.87612E-08
20	-0.08554	0.000551	0.009211	1.110793	0.000237	0.000623194	3.78175E-07	-5.71044E-08
Mean	-0.41253	0.029707	0.100512	1.110556		0.000622816		-1.3077E-07

Using the second method, the CO_2 flux is $\overline{\rho_d}\overline{w's_c'} = 1.118 \times (-1.3077 \times 10^{-7}) = -1.4620 \times 10^{-7}$ kg m^{-2} s^{-1}. This is nearly the same as the value, -1.4367×10^{-7} kg m^{-2} s^{-1}, from method 1. This is because both methods are based on the ideal gas law and the Dalton's law of partial pressures. It also shows that omission of higher order terms in the Taylor expansion used by the first method gives acceptable accuracy.

9.4 Without correction for the density effects, is the evaporation rate measured with open-path eddy covariance in unstable conditions biased high or low? What about the measurement made with closed-path eddy covariance?

In unstable conditions, the dry air density fluctuations are generally negative correlated with w. Thus, the evaporation rate measured with an open-path system is biased low according to Original Equation 9.4.

When using a closed-path analyzer, you can eliminate the density effect associated with temperature. However, the density effect due to water vapour is still present, which, according to Original Equation 9.19, would cause the measurements to still be biased low.

9.5 An open-path eddy covariance system in the middle of a large parking lot made of concrete yields a $\overline{w'\rho'_v}$ value of -0.016 g m^{-2} s^{-1} at midday (Ham and Heilman 2003). Does the measurement indicate that condensation is occurring on the concrete surface?

At midday, the skin temperature of a concrete surface is usually higher than the air temperature, in which condition condensation usually does not occur. The negative measurement results from the density-effect caused by temperature fluctuations.

9.6 Calculate the true CO_2 and water vapor flux using these half-hourly mean Reynolds statistics obtained with an open-path eddy covariance system: $\overline{w'\rho'_c} = -1.22$ mg m^{-2} s^{-1}, $\overline{w'T'} = 0.320$ K m s^{-1}, $\overline{w'\rho'_v} = 0.109$ g m^{-2} s^{-1}, $\overline{\rho}_c = 719.3$ mg m^{-3}, $\overline{\rho}_v = 7.94$ g m^{-3}, $\overline{T} = 19.2$ °C, and $\overline{p} =$ 997.2 hPa.

To solve this problem, we must first find the mean CO_2 and water vapor mixing ratios, given by:

$$\overline{s_v} = \frac{\overline{\rho_v}}{\overline{\rho_d}}, \qquad \overline{s_c} = \frac{\overline{\rho_c}}{\overline{\rho_d}}$$

where the mean dry air density is found using Original Equation 9.12:

$$\begin{aligned}
\overline{\rho_d} &= \frac{M_d\overline{p}}{R\overline{T}} - \mu\overline{\rho_v} \\
&= \frac{(0.029 \text{ kg mol}^{-1})(997.2 \times 10^2 \text{ kg m}^{-3})}{(8.314\text{J mol}^{-1}\text{ K}^{-1})(292.35 \text{ K})} - \frac{29}{18}(0.00794 \text{ kg m}^{-3}) \\
&= 1.18 \text{ kg m}^{-3}
\end{aligned}$$

The mean CO_2 and water vapor mixing ratios are 609.6×10^{-6} and 6.73×10^{-3}, respectively.

True CO_2 flux can be found using Original Equation 9.18:

$$\begin{aligned}
\text{NEE} &\simeq \overline{w'\rho_c} + \overline{\rho_c}(1 + \mu\overline{s_v})(\frac{\overline{w'T'}}{\overline{T}}) + \mu\overline{s_c}(\overline{w'\rho'_v}) \\
&= -1.22 \text{ mg m}^{-3}\text{s}^{-1} + 719.2 \text{ mg m}^{-3} \times (1 + \frac{29}{18} \times 6.73 \times 10^{-3})(\frac{0.320 \text{ K m s}^{-1}}{292.35 \text{ K}}) \\
&+ \frac{29}{18} \times 606.6 \times 10^{-6} \times 109 \text{ mg m}^{-2}\text{ s}^{-1} \\
&= -0.32 \text{ mg m}^{-2} \text{ s}^{-1}
\end{aligned}$$

Water vapor flux is given by Original Equation 9.19:

$$\text{E} = (1 + \mu\overline{s_v})[\overline{w'\rho'_v} + \overline{\rho_v}(\frac{\overline{w'T'}}{\overline{T}})] = 0.119 \text{ kg m}^{-2}\text{ s}^{-1}$$

9.7 The annual mean sensible and latent heat flux are 100.3 and 14.9 W m^{-2}, respectively, in a semiarid plantation forest. Determine the density correction to the annual mean CO_2 flux measured with an open-path eddy covariance system. Use the data given in Problem 9.6 for the mean water vapor density, CO_2 density, air temperature and air pressure. Now assume that the mean CO_2 density $\overline{\rho}_c$ has been underestimated by 10%. How large is the CO_2 flux bias error caused by propagation of the $\overline{\rho}_c$ measurement error through the density correction procedure? (For your reference, the annual NEE of the forest is -2.3 tC ha^{-1} y^{-1}.)

The eddy flux of sensible heat(F_h) and latent heat(λE) are:

$$F_h = \overline{\rho}_d c_p(\overline{w'T'})$$
$$\lambda E = \lambda(1 + \mu\overline{s}_v)[\overline{w'\rho_v'} + \overline{\rho}_v \frac{\overline{w'T'}}{\overline{T}}]$$

Combining the above two equations, we obtain:

$$\overline{w'T'} = \frac{F_h}{\overline{\rho}_d c_p}$$
$$\overline{w'\rho_v'} = \frac{\lambda E}{\lambda(1 + \mu\overline{s}_v)} - \overline{s}_v \frac{F_h}{c_p \overline{T}}$$

Therefore, if $\overline{\rho}_c$ is underestimated by 10%, the temperature correction term will be underestimated by:

$$\begin{aligned}
10\%\overline{\rho}_c(1 + \mu\overline{s}_v)\frac{\overline{w'T'}}{\overline{T}} &= 10\%\frac{F_h\overline{\rho}_c(1 + \mu\overline{s}_v)}{\overline{\rho}_d c_p \overline{T}} \\
&= 10\%\frac{100.3 \times 719.3 \times 10^{-6} \times (1 + 1.61 \times 6.73 \times 10^{-3})}{1.18 \times 1004 \times (19.2 + 273.15)} \\
&= 2.0 \times 10^{-8}\ \mathrm{kg\ m^{-2}\,s^{-1}} \\
&= 6.3\ \mathrm{t\,CO_2\,ha^{-1}\,y^{-1}} \\
&= 1.7\ \mathrm{t\,C\,ha^{-1}\,y^{-1}}
\end{aligned}$$

Similarly, the water vapor correction term will be underestimated by:

$$\begin{aligned}
10\%\mu\overline{s}_c(\overline{w'\rho_v'}) &= 10\%\mu\frac{\overline{\rho}_c}{\overline{\rho}_d}\left(\frac{\lambda E}{\lambda(1 + \mu\overline{s}_v)} - \overline{s}_v \frac{F_h}{c_p \overline{T}}\right) \\
&= 10\%\frac{1.61 \times 719.3 \times 10^{-6}}{1.18} \times \left(\frac{14.9}{2.45 \times 10^6 \times (1 + 1.61 \times 6.73 \times 10^{-3})}\right. \\
&\quad \left. - \frac{6.73 \times 10^{-3} \times 100.3}{1004 \times (19.2 + 273.15)}\right) \\
&= 3.9 \times 10^{-10}\ \mathrm{kg\,m^{-2}\,s^{-1}} \\
&= 0.1\ \mathrm{t\,C\,ha^{-1}\,y^{-1}}
\end{aligned}$$

Therefore, a 10% underestimation of $\overline{\rho}_c$ will cause the annual NEE to be biased by 1.8 t C ha^{-1} y^{-1}, an error comparable in magnitude to the true NEE.

9.8 The eddy covariance sensible heat flux has a typical random error of 10 $\mathrm{W\,m^{-2}}$ in daylight hours. How large is the CO_2 flux uncertainty caused by the density correction procedure if the measurement is made with an open-path eddy covariance system? The CO_2 flux signal of an unpolluted lake is on the order of 1 $\mu\mathrm{mol\,m^{-2}\,s^{-1}}$. Can the eddy covariance system resolve the CO_2 flux signal at hourly intervals?

Using the equation established in Problem 9.7, the uncertainty caused by the sensible heat flux random error is:

$$\begin{aligned}\frac{F_h \overline{s}_c(1+\mu\overline{s}_v)}{c_p\overline{T}} &\simeq \frac{10\times 6.3\times 10^{-4}\times(1+1.6\times 6.9\times 10^{-3})}{1004\times(19.2+273.15)}\\ &= 2.0\times 10^{-8}\ \mathrm{kg\,m^{-2}\,s^{-1}}\\ &= 0.45\ \mu\mathrm{mol\,m^{-2}\,s^{-1}}\end{aligned}$$

Because this CO_2 flux random error is similar in magnitude to the true flux signal, it is difficult to resolve the hourly flux with an open-path eddy covariance system.

9.9 Using the information provided in Problem 9.6, estimate the amount of density correction needed for the N_2O flux measured with an open-path eddy covariance system. How large is the correction in comparison with a typical cropland N_2O flux of 0.3 $\mathrm{nmol\,m^{-2}\,s^{-1}}$? How large is the N_2O flux uncertainty associated with a random error of 10 $\mathrm{W\,m^{-2}}$ in the sensible heat flux?

Assuming that the mass density of N_2O ($\overline{\rho}_N$) is 4.4×10^{-7} kg m^{-3}, the density correction is:

$$\begin{aligned}\overline{\rho}_N(1+\mu\overline{s}_v)(\frac{\overline{w'T'}}{\overline{T}})+\frac{\mu\overline{\rho}_N}{\overline{\rho}_d}(\overline{w'\rho_v'}) &= 5.5\times 10^{-10}\ \mathrm{kg\ m^{-2}\ s^{-1}}\\ &= 12.6\ \mathrm{nmol\ m^{-2}\ s^{-1}}\end{aligned}$$

The density correction is 46 times of the typical cropland N_2O flux.

The uncertainty caused by the random error of sensible heat flux is:

$$\begin{aligned}\overline{\rho}_N(1+\mu\overline{s}_v)(\frac{\overline{w'T'}}{\overline{T}}) &\simeq 4.4\times 10^{-7}(1+1.61\times 6\times 10^{-3})\frac{0.0083}{293}\\ &= 1.3\times 10^{-11}\ \mathrm{kg\,m^{-2}\,s^{-1}}\\ &= 0.3\ \mathrm{nmol\,m^{-2}\,s^{-2}}\end{aligned}$$

which is comparable in magnitude to the expected true flux signal.

9.10 Derive density correction equations for closed-path eddy covariance. Use the equations to calculate the true CO_2 and water vapor flux using these half-hourly mean Reynolds statistics:

$\overline{w'\rho_c'} = -0.45\ \mathrm{mg\,m^{-2}\,s^{-1}}$, $\overline{w'\rho_v'} = 0.169\ \mathrm{g\,m^{-2}\,s^{-1}}$, $\overline{w'T'} = 0.205\ \mathrm{K\,m\,s^{-1}}$, $\overline{\rho}_c = 708.3\ \mathrm{mg\,m^{-3}}$, $\overline{\rho}_v = 20.16\ \mathrm{g\,m^{-3}}$, $\overline{T} = 17.7\ °\mathrm{C}$, and $\overline{p} = 950.8$ hPa.

For a closed-path system, the density correction terms associated with temperature are eliminated. Based on the dry air density:

$$\overline{\rho}_d = \frac{M_d\overline{p}}{R\overline{T}} - \mu\overline{\rho}_v = 1.108\ \mathrm{kg\,m^{-3}}$$

For true CO_2 flux, the formula is simplified to:

$$\begin{aligned}\mathrm{NEE} &= \overline{w'\rho_c'} + \mu\overline{s}_c(\overline{w'\rho_v'}) \\ &= -0.45 + 1.61 \times \frac{708.3}{1108} \times 0.169 \\ &= -0.28\ \mathrm{mg\,m^{-2}\,s^{-1}}\end{aligned}$$

and for true water vapor flux, it is:

$$E = (1 + \mu\overline{s}_v)\overline{w'\rho_v'} = 0.174\ \mathrm{kg\ \ m^{-2}\,s^{-1}}$$

9.11 In a ship-borne experimental campaign in the North Atlantic, the sea-air CO_2 flux was measured with two eddy covariance systems. One was a standard closed-path system, and the other was a closed-path system fitted with a Nafion drier upstream of its gas analyzer. The latent flux measured with the two systems was 48 and 1.2 $\mathrm{W\,m^{-2}}$, respectively. During the campaign, the air temperature was 12.9 °C, the air pressure was 998.9 hPa, the water vapor pressure was 8.96 hPa, and the CO_2 molar mixing ratio was 380.2 ppm. Compute the density corrections to the CO_2 flux associated with the water vapor density effect. Are the density corrections larger or smaller in magnitude than the true ocean surface CO_2 flux of $-3.1\ \mathrm{mol\,m^{-2}\,y^{-1}}$?

The CO_2 flux from a closed-path system is given by:

$$\mathrm{NEE} = \overline{w'\rho_c'} + \mu\overline{s}_c(\overline{w'\rho_v'})$$

Converting the latent flux of 48 $\mathrm{W\,m^{-2}}$ to water vapor flux, we have:

$$E = \frac{48\ \mathrm{W\,m^{-2}}}{2466\ \mathrm{J\,g^{-1}}} = 0.0195\ \mathrm{g\,m^{-2}\,s^{-1}}$$

We know that $\mu = \frac{M_d}{M_v} = \frac{0.029}{0.018} = 1.61$.

We are given the CO_2 molar mixing ratio $\overline{\chi}_c = 380.2$ ppm. We must convert to the mass mixing ratio $\overline{s}_c$:

$$\overline{s}_c = \frac{380.2}{1,000,000} \times \frac{0.044}{0.029} = 0.00057686\ \mathrm{g\,g^{-1}} = 576.9\ \mathrm{\mu g\,g^{-1}}$$

The water vapor density effect is therefore:

$$\begin{aligned}\mu\overline{s}_c(\overline{w'\rho_v'}) &= 1.61 \times 576.9\ \mu\text{g g}^{-1} \times 0.0195\ \text{g m}^{-2}\,\text{s}^{-1}\\ &= 18.1\ \mu\text{g m}^{-2}\,\text{s}^{-1}\\ &= 13\ \text{mol m}^{-2}\,\text{y}^{-1}\end{aligned}$$

which is much larger than the true ocean surface flux.

Use of a Nafion drier reduces the latent heat flux substantially to 1.2 W m^{-2}. The corresponding density effect is 0.33 $\text{mol m}^{-2}\,\text{y}^{-1}$, which is an order of magnitude smaller than the true flux signal.

9.12 Using the information provided in Original Figure 9.3 and the flux-gradient relation, determine the true CO_2 flux. The eddy diffusivity is 0.3 $\text{m}^2\,\text{s}^{-1}$.

In the scenario of Original Figure 9.3, the air is completely dry so omitting water vapor effect from Original Equation 9.23, we get:

$$F_c = -K_c[\frac{\partial\overline{\rho}_c}{\partial z} + \frac{\overline{\rho}_c}{\overline{T}}\frac{\partial\overline{T}}{\partial z}].$$

From the text, $\partial\overline{\rho}_c/\partial z = 2.1$ mg m^{-4}. Estimated from the figure, $\overline{\rho}_c = 670$ mg m^{-3}, $\overline{T} = 45°\text{C}$ $= 318.15$ K and $\partial\overline{T}/\partial z = -1$ K m^{-1}. The true flux thus calculated is about 1.77×10^{-3} $\text{mg m}^{-2}\,\text{s}^{-1}$.

An alternative approach is to use Original Equation 9.21:

$$F_c = -K_c\,\overline{\rho}_d\frac{\partial\overline{s}_c}{\partial z}$$

According to Original Figure 9.3, $\partial\overline{s}_c/\partial z = 0$, so the true flux is $F_c = 0$ $\text{mg m}^{-2}\,\text{s}^{-1}$.

9.13 Some people say that there are no density effects on the water vapor flux if it is measured with a close-path eddy covariance system. Are they correct?

No, they are incorrect. The only way to get rid of the density effect due to water vapor fluctuations is to dry the air in advance, which cannot be done when water vapor is the target gas. A close-path system can only remove the temperature effect.

9.14* In a field experiment deploying the flux-gradient method, air is drawn continuously from intakes at two heights above the surface, and measurements of the CO_2 and water vapor densities are made in an alternate sequence by a CO_2/H_2O dual gas analyzer at a common temperature T_c and a common pressure p_c. Simultaneous measurement is also made of air temperature at the same heights. Derive an expression for the determination of the true CO_2 flux from these measurements. (Assume that the eddy diffusivity is known.)

As both CO_2 and water vapor density are measured at a common temperature, the temperature gradient effect is removed. Original Equation 9.24 is then reduced to

$$F_c = -K_c[\frac{\partial \overline{\rho}_c}{\partial z} + \mu \overline{s}_c \frac{\partial \overline{\rho}_v}{\partial z}]. \tag{9.1}$$

Suppose that the densities of CO_2 and water vapor at height z_1 are $\rho_{c,1}$ and $\rho_{v,1}$, respectively, and those at height z_2 are $\rho_{c,2}$ and $\rho_{v,2}$. The mean CO_2 mixing ratio is given by:

$$\overline{s}_c = \frac{0.5(\overline{\rho}_{c,1} + \overline{\rho}_{c,2})}{\overline{\rho}_d}$$

where the mean dry air density $\overline{\rho}_d$ is given by Original Equation 9.14:

$$\overline{\rho}_d = \frac{M_d\, p_c}{RT_c} - 0.5\mu(\overline{\rho}_{v,1} + \overline{\rho}_{v,1})$$

The vertical derivatives in Equation 9.1 are approximated by their finite differences:

$$\frac{\partial \overline{\rho}_c}{\partial z} = \frac{\overline{\rho}_{c,2} - \overline{\rho}_{c,1}}{z_2 - z_1}$$

$$\frac{\partial \overline{\rho}_v}{\partial z} = \frac{\overline{\rho}_{v,2} - \overline{\rho}_{v,1}}{z_2 - z_1}$$

In this configuration, we do not need ambient air temperature measurements.

9.15* In a chamber blank test, you place a chamber with a pressure equilibration port (Original Figure 9.4) on a surface that evaporates water but does not emit or absorb CO_2. Show that the CO_2 mixing ratio in the chamber does not change with time even though the addition of water vapor will force some air to leak out of the chamber.

This conclusion can be reached by a thought experiment. Initially, the chamber is airtight. At the beginning, there is no water vapor in the chamber volume. The total pressure in the chamber, $p_d + p_c$, is equal to the ambient pressure, where p_d and p_c are partial pressures of dry air and and carbon dioxide, respectively. After some finite time, water vapor has built up in the chamber, causing the total pressure in the chamber to increase to $p_e + p_d + p_c$, where p_v is the partial pressure of water vapor. Now we open the equilibrium port. The over-pressure will push some air out of the chamber, causing p_d and p_c to go down until the total pressure is the same as the ambient pressure. We assume that the chamber air is well mixed. Because p_d and p_c are reduced by the same relative amount, the ratio p_c/p_d, or the CO_2 molar mixing ratio, remains unchanged from its initial value.

9.16 A closed chamber is equipped with a pressure equilibration port so that the pressure inside the chamber is maintained at the same level as the ambient pressure, which can be considered as a constant over the measurement interval. Derive from Original Equations 9.12 and 9.25 the density correction Original Equation 9.27 for the chamber CO_2 flux.

We can obtain ρ_c from Original Equation 2.10:

$$s_c = \frac{\rho_c}{\rho_d} \rightarrow s_c \rho_d = \rho_c$$

The derivative of the above equation is:

$$\frac{\partial \rho_d}{\partial t} s_c + \rho_d \frac{\partial s_c}{\partial t} = \frac{\partial \rho_c}{\partial t}$$

So we get:

$$\rho_d \frac{\partial s_c}{\partial t} = \frac{\partial \rho_c}{\partial t} - \frac{\partial \rho_d}{\partial t} s_c$$

To obtain Original Equation 9.27:

$$F_c = \frac{V}{A}\left[\frac{\partial \rho_c}{\partial t} + (1+\mu s_v)\frac{\rho_c}{T}\frac{\partial T}{\partial t} + \mu s_c \frac{\partial \rho_v}{\partial t}\right]$$

We need to substitute $\rho_d \frac{\partial s_c}{\partial t}$ into Original Equation 9.25, which leads to:

$$F_c = \frac{V}{A}\left[\frac{\partial \rho_c}{\partial t} - \frac{\partial \rho_d}{\partial t} s_c\right]$$

Substitute in the time derivative of ρ_d, which can be obtained from the derivative of Original Equation 9.12:

$$\frac{\partial \rho_d}{\partial t} = -\frac{M_d\, p}{RT^2}\frac{\partial T}{\partial t} - \mu \frac{\partial \rho_v}{\partial t}$$

In this step we neglect the pressure change with time, which is usually small.

With this, we further simplify Original Equation 9.25 as:

$$F_c = \frac{V}{A}\left[\frac{\partial \rho_c}{\partial t} + s_c \frac{M_d p}{RT^2}\frac{\partial T}{\partial t} + \mu\, s_c \frac{\partial \rho_v}{\partial t}\right]$$

We note that:

$$s_c \frac{M_d p}{RT^2} = \frac{\rho_c}{\rho_d}\,\frac{M_d p_d (1 + p_v/p_d)}{RT}\,\frac{1}{T} = (1 + p_v/p_d)\frac{\rho_c}{T} = (1+\mu s_v)\frac{\rho_c}{T}$$

Combining that last two equations above, we obtain Original Equation 9.27.

9.17 The density correction method assumes that all the variables used for the correction are measured perfectly. In practice, measurement errors are unavoidable, and can propagate through the density correction procedure to degrade the quality of trace gas flux data. Assuming that the mean CO_2 density and the mixing ratio are biased low by 5%, estimate the bias error in the annual cumulative CO_2 flux measured with open-path eddy covariance. (The annual mean sensible and latent heat fluxes are 50 and 70 $\mathrm{W\,m^{-2}}$, respectively.)

To get a baseline correction value, use the values from Problem 9.6 for $\overline{\rho}_c$, $\overline{\rho}_v$, and $\overline{T}$ =15° C Use Original Equation 3.60 to get $\overline{w'T'}$:

$$\overline{w'T'} = \frac{F_h}{\rho_d c_p} = \frac{50}{(1.225)(1004)} = 0.0406\ \mathrm{K\,m\ \ s^{-1}}$$

Next convert latent heat flux to water vapor flux:

$$E = \frac{70}{2466} = 0.0284\ \mathrm{g\,m^{-2}\,s^{-1}}$$

Then use $\overline{w'T'}$ and Original Equation 9.19 to solve for $\overline{w'\rho_v'}$:

$$\begin{aligned}\overline{w'\rho_v'} &= \frac{E}{(1+\mu\overline{s}_v)} - \overline{\rho}_v\left(\frac{\overline{w'T'}}{\overline{T}}\right)\\ &= \frac{0.0284}{(1+\frac{0.029}{0.018}\times 0.00648)} - 7.94\left(\frac{0.0406}{288.15}\right)\\ &= 0.02697\ \mathrm{g\,m^{-2}\,s^{-1}}\end{aligned}$$

Then the NEE correction term from Original Equation 9.18 is:

$$\begin{aligned}\text{Correction} &= \overline{\rho}_c(1+\mu\overline{s}_v)\left(\frac{\overline{w'T'}}{\overline{T}}\right) + \mu\overline{s}_c(\overline{w'\rho_v'})\\ &= (719.3)(1+\frac{0.029}{0.018}(0.00648))\left(\frac{0.0406}{288.15}\right) + \frac{0.029}{0.018}(0.000587)(0.02697)\\ &= 0.1028\ \mathrm{g\ m^{-2}\ s^{-1}}\end{aligned}$$

If mean CO_2 density and mixing ratio are biased low by 5 percent, you get:

$$\begin{aligned}\text{Correction} &= (0.95)(719.3)(1+\frac{0.029}{0.018}(0.00648))\left(\frac{0.0406}{288.15}\right) + (0.95)\frac{0.029}{0.018}(0.000587)(0.02697)\\ &= 0.0977\ \mathrm{g\,m^{-2}\,s^{-1}}\end{aligned}$$

These two flux values differ by 0.0051 $\mathrm{mg\,m^{-2}\,s^{-1}}$, which is equivalent to an annual bias error of 0.44 $\mathrm{t\,C\,ha^{-1}\,y^{-1}}$.

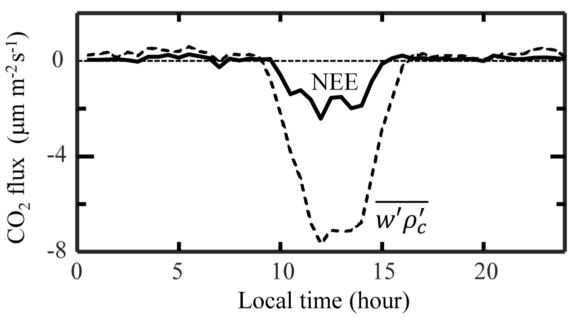


Figure 9.1: (Original Figure 9.5) Diurnal composite of the CO_2 flux observed with open-path eddy covariance over a desert ecosystem in Northwest China in a winter dormant season: dashed line, covariance between the vertical velocity and the CO_2 density; Solid line, density-corrected CO_2 flux (Original Equation 9.18). Data source: Wang et al. (2016).

9.18 Open-path eddy covariance systems often register physiologically unreasonable CO_2 uptake signals during the off-season in extreme cold environments (Original Figure 9.5). A hypothesized cause of this artificial flux is that the CO_2 analyzer itself generates heat, but the density correction procedure uses temperature fluctuations measured outside the analyzer's optical path, and as such fails to fully correct the density effects. How large would an additional heat flux due to sensor self-heating be required to correct the midday CO_2 shown in Figure 9.1 to zero?

As shown in Original Figure 9.5, a carbon dioxide flux of about $-2\ \mu\mathrm{mol\,m^{-2}\,s^{-1}}$ (-0.088 $\mathrm{mg\,m^{-2}\,s^{-1}}$) is recorded as the density correction procedure fails to fully correct the density effects. An additional temperature correction would need to be:

$$\overline{\rho}_c(1+\mu\overline{s}_v)\frac{\overline{w'T'}}{\overline{T}} = 0.088\ \mathrm{mg\,m^{-2}\,s^{-1}}$$

to force the flux to zero. Using the ratio of molecular mass of dry air to that of water vapor $\mu = 1.61$, carbon dioxide mass density $\overline{\rho}_c = 800\ \mathrm{mg\,m^{-3}}$, air temperature $\overline{T} = 273$ K, and water vapor mixing ratio $\overline{s}_v = 5\ \mathrm{g\,kg^{-1}}$, we estimate that:

$$\overline{w'T'} = 0.030\ \mathrm{K\,m\,s^{-1}}$$

which is equivalent to an additional heat flux of about 36 $\mathrm{W\,m^{-2}}$.

9.19 The sensible and latent heat fluxes are 240.1 and 108.7 $\mathrm{W\,m^{-2}}$, respectively, the air temperature is 23.4°C, the atmospheric pressure is 997.4 hPa, the water vapor and carbon dioxide mixing ratios are 21.2 $\mathrm{g\,kg^{-1}}$ and 610.7 $\mathrm{mg\,kg^{-1}}$, respectively, and the covariance $\overline{w'\rho_c'}$ measured with an open-path eddy covariance system is $-2.09\ \mathrm{mg\,m^{-2}s^{-1}}$. Find the true net ecosystem CO_2 exchange.

Use Original Equation 3.60 to get $\overline{w'T'}$:

$$\overline{w'T'} = \frac{F_h}{\rho_d c_p} = \frac{240.1}{(1.225)(1004)} = 0.1952 \text{ K m s}^{-1}$$

Next convert latent heat flux to water vapor flux:

$$E = \frac{108.7}{2466} = 0.044 \text{ g m}^{-2}\text{ s}^{-1}$$

Then use $\overline{w'T'}$ and Original Equation 9.19 to solve for $\overline{w'\rho_v'}$, noting $\overline{\rho}_v = \overline{s}_v\overline{\rho}_d = 21.2 \text{ g kg}^{-1} \times 1.225 \text{ kg m}^{-3} = 25.97 \text{ g m}^{-3}$:

$$\begin{aligned}\overline{w'\rho_v'} &= \frac{E}{(1+\mu\overline{s}_v)} - \overline{\rho}_v\left(\frac{\overline{w'T'}}{\overline{T}}\right)\\ &= \frac{0.044}{(1+\frac{0.029}{0.018}\times 21.2)} - 25.97\left(\frac{0.1952}{296.55}\right)\\ &= 0.0255 \text{ g m}^{-2}\text{ s}^{-1}\end{aligned}$$

Plug all of these into Original Equation 9.18 to get NEE (ignoring the pressure term because it is negligible), noting $\overline{\rho}_c = \overline{s}_c\overline{\rho}_d = 610.7 \text{ mg kg}^{-1} \times 1.225 \text{ kg m}^{-3} = 748 \text{ mg m}^{-3}$

$$\begin{aligned}\text{NEE} &= \overline{w'\rho_c'} + \overline{\rho}_c(1+\mu\overline{s}_v)\left(\frac{\overline{w'T'}}{\overline{T}}\right) + \mu\overline{s}_c(\overline{w'\rho_v'})\\ &= -2.09 + 748 \times (1 + \frac{0.029}{0.018}\times 0.0212)\left(\frac{0.1952}{296.55}\right) + \frac{0.029}{0.018}(0.6107)(0.0255)\\ &= -1.56 \text{ mg m}^{-2}\text{ s}^{-1}\end{aligned}$$

Chapter 10

Energy Balance, Evaporation, and Surface Temperature

10.1 Calculate the leaf-scale sensible and latent heat flux using the following data: boundary layer resistance = 16 $\mathrm{s\,m^{-1}}$, stomatal resistance = 50 $\mathrm{s\,m^{-1}}$, leaf temperature = 16.3°C, air temperature = 14.6°C, and water vapor mixing ratio =17.8 $\mathrm{g\,kg^{-1}}$.

Sensible heat can be calculated using Original Equation 10.2:

$$\begin{aligned} H_l &= \rho_d c_p \frac{T_l - T_a}{r_b} \\ &= 1.177 \text{ kg m}^{-3} \times 1004 \text{ J kg}^{-1}\,\text{K}^{-1} \times \frac{16.3 - 14.6°\text{C}}{16 \text{ s m}^{-1}} = 125.6 \text{ W m}^{-2} \end{aligned}$$

Water vapor flux can be calculated using Original Equation 10.4:

$$\begin{aligned} E_l &= \rho_d \frac{s_v^* - s_v}{r_s + r_b} = \rho_d \frac{0.621 e_v^*/p_d - s_v}{r_s + r_b} \\ &= 1.177 \text{ kg m}^{-3} \times \frac{0.621 \times 16.62 \text{ hPa}/1013 \text{ hPa} - 0.0178 \text{ kg kg}^{-1}}{50 \text{ s m}^{-1} + 16 \text{ s m}^{-1}} = 0.14 \text{ g m}^{-2}\,\text{s}^{-1} \end{aligned}$$

Latent heat flux is: $\lambda E_l = 2466 \text{ J g}^{-1} \times 0.14 \text{ g m}^{-2}\text{s}^{-1} = 345.2 \text{ W m}^{-2}$

10.2 What is the vapor pressure deficit if air temperature is 12.1°C and relative humidity is 48.5%?

Vapor pressure deficit D can be calculated as:

$$\begin{aligned} D &= e_v^*(T_a) - e_v \\ &= e_v^*(T_a) \times (1 - \text{RH}) \\ &= 14.11 \text{ hPa} \times (1 - 48.5\%) = 7.27 \text{ hPa} \end{aligned}$$

10.3 When exposed to solar radiation, thermometers can significantly overestimate air temperature. Determine measurement error for thermometers of 20 μm in diameter (fine-wire thermocouples) and 0.5 cm in diameter (mercury thermometers). Do the calculation for two levels of wind speed (0.1 and 10 $\mathrm{m\,s^{-1}}$) and with the thermometers in the shade (net radiation 2 $\mathrm{W\,m^{-2}}$) and exposed to the Sun (net radiation 50 $\mathrm{W\,m^{-2}}$). Can you suggest preventive measures to minimize the bias error? (Hint: These thermometers satisfy the energy balance equation for non-evaporating leaves.)

Treat the thermometers as non-evaporating leaves, which satisfy the energy balance equation:

$$R_{n,l} = H_l = \rho_d c_p \frac{T_l - T_a}{r_b}$$

The difference between the "leaf temperature" and air temperature $T_l - T_a$, which is the error of these thermometers, is given by:

$$T_l - T_a = \frac{R_{n,l} r_b}{\rho_d c_p}$$

By using the common parameterization of boundary layer resistance (Original Equation 10.8), we have:

$$T_l - T_a = \frac{R_{n,l}}{\rho_d c_p} \frac{1}{C_l} \sqrt{\frac{d_l}{u_l}}$$

where d_l is the diameter of the thermometer and u_l is wind speed. Using the above equation, we can calculate errors under different conditions, which are showed in Table 10.1. As can be seen from the table, the error is smaller when wind is stronger, the diameter is smaller and without direct solar radiation. The smallest error of 0.0002 °C occurs when $d_l = 20$ μm and $u_l = 10$ m s^{-1}. We recommend to use a thermometer with a smaller diameter and put it in a shaded and well ventilated location, in order to minimize bias errors.

Table 10.1: Errors of temperature measurement (°C)

		d_l=20 μm	d_l=0.5 cm
u_l=0.1 m s^{-1}	sun-lit	0.0575	0.9090
	shaded	0.0023	0.0364
u_l=10 m s^{-1}	sun-lit	0.0057	0.0909
	shaded	0.0002	0.0036

10.4 Humans are homeotherms whose deep body temperature is regulated at about 37°C. In order to dissipate metabolic heat generated internally, they must keep the skin temperature lower than 35°C. Otherwise their deep body temperature will increase, leading to heat stroke or even death. Calculate the skin temperature of a naked and perspiring human body in hot and humid conditions (air temperature 38.3°C and relative humidity 54.5%). Now repeat the calculation by increasing the air temperature by 2°C and 4°C to simulate heat stress and human health effects caused by the urban heat island and by global warming, respectively.

Under which scenario(s) is the threshold temperature of 35°C exceeded? (Hint: the perspiring human body can be considered as a wet bulb.)

To calculate wet bulb temperature, we can use Original Equation 10.12:

$$T_w = T_a - \frac{D}{\Delta + \gamma}$$

Vapor pressure deficit D can be calculated as:

$$\begin{aligned} D &= e_v^*(T_a) - e_v \\ &= e_v^*(T_a) \times (1 - \mathrm{RH}) \\ &= 67.42 \text{ hPa} \times (1 - 54.5\%) = 30.68 \text{ hPa} \end{aligned}$$

where 67.42 hPa is the saturation vapor pressure at 38.3°C. To linearize $\partial e_v^* / \partial T_a$ at 38.3°C, we can calculate the slope from 38.2 to 38.4°C:

$$\Delta \simeq \frac{67.71 - 66.98}{38.4 - 38.2} = 3.65 \text{ hPa}\,\mathrm{K}^{-1}$$

Therefore:

$$T_w = 38.3 - \frac{30.68}{3.65 + 0.66} = 31.2°\mathrm{C}$$

When air temperature increases by 2°C:

$$\begin{aligned} D &= 75.04 \text{ hPa} \times (1 - 54.5\%) = 34.14 \text{ hPa} \\ \Delta &\simeq \frac{75.36 - 74.56}{40.4 - 40.2} = 4 \text{ hPa}\,\mathrm{K}^{-1} \\ T_w &= 40.3 - \frac{34.14}{4 + 0.66} = 33.0° \mathrm{C} < 35.0° \mathrm{C} \end{aligned}$$

In this scenario, the threshold temperature is not exceeded.

When air temperature increases by 4°C:

$$\begin{aligned} D &= 83.31 \text{ hPa} \times (1 - 54.5\%) = 37.9 \text{ hPa} \\ \Delta &\simeq \frac{83.31 - 84.19}{42.20 - 42.4} = 4.4 \text{ hPa K}^{-1} \\ T_w &= 42.3 - \frac{37.9}{4.4 + 0.66} = 35.6° \mathrm{C} > 35.0° \mathrm{C} \end{aligned}$$

In this scenario, the threshold temperature is exceeded.

10.5 Derive the Penman-Monteith Equation 10.17 from Equations 10.13 to 10.15.

Substitute $s_v = 0.621e_v/p_d$ and the Taylor approximation of $e_v^*(T_s)$ into the evaporation flux equation:

$$\begin{aligned}
E &= \rho_d \frac{s_v^* - s_v}{r_c + r_a} \\
&= \rho_d \frac{\frac{0.621}{p_d}(e_v^*(T_a) - e_v + \Delta(T_s - T_a))}{r_c + r_a} \\
&= \frac{0.621\rho_d}{p_d} \frac{D + \Delta(T_s - T_a)}{r_c + r_a},
\end{aligned}$$

where $D = e_v^*(T_a) - e_v$ is the vapor pressure deficit. It is then easy to express sensible heat H in terms of E (by eliminating $T_s - T_a$).

$$\begin{aligned}
H &= \rho_d c_p \frac{T_s - T_a}{r_a} \\
&= \frac{\rho_d c_p}{r_a} \frac{p_d(r_c + r_a)}{0.621\rho_d \Delta} E - \frac{\rho_d c_p D}{r_a \Delta} \\
&= \frac{c_p p_d (r_c + r_a)}{0.621 r_a \Delta} E - \frac{\rho_d c_p D}{r_a \Delta} \\
&= \gamma \frac{(r_c + r_a)}{r_a \Delta} \lambda E - \frac{\rho_d c_p D}{r_a \Delta}
\end{aligned}$$

Substitute this relation into the energy balance equation:

$$\begin{aligned}
R_n - G &= H + \lambda E \\
R_n - G &= (1 + \gamma \frac{(r_c + r_a)}{r_a \Delta}) \lambda E - \frac{\rho_d c_p D}{r_a \Delta}
\end{aligned}$$

Then it follows that:

$$\begin{aligned}
\lambda E &= \frac{\Delta(R_n - G) + \rho_d c_p D/r_a}{\Delta + \gamma(r_a + r_c)/r_a} \\
E &= \frac{1}{\lambda} \frac{\Delta(R_n - G) + \rho_d c_p D/r_a}{\Delta + \gamma(r_a + r_c)/r_a}.
\end{aligned}$$

10.6 The available energy is 293.4 $\mathrm{W\,m^{-2}}$, the vapor pressure deficit is 7.1 hPa, the aerodynamic resistance is 44 $\mathrm{s\,m^{-1}}$, and the air temperature is 15.2°C. Find the evaporation rate of a dry surface (canopy resistance 109 $\mathrm{s\,m^{-1}}$) and a wet surface (canopy resistance 0 $\mathrm{s\,m^{-1}}$) using the Penman-Monteith equation.

At air temperature of 15.2°C, $\Delta \simeq 1.1\mathrm{hPa\,K^{-1}}$. Take $\rho_d = 1.25\ \mathrm{kg\,m^{-3}}$, $c_p = 1004\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $\gamma = 0.66\ \mathrm{hPa\,K^{-1}}$ and $\lambda = 2.5 \times 10^3\ \mathrm{J\,g^{-1}}$. We are also given $R_n - G = 293.4\ \mathrm{W\,m^{-2}}$, $D = 7.1$ hPa and $r_a = 44\ \mathrm{s\,m^{-1}}$. Substituting these values into Original Equation 10.17, we have:

For a dry surface with $r_c = 109\ \mathrm{s\,m^{-1}}$: $E = 0.062\ \mathrm{g\,m^{-2}\,s^{-1}}$.

For a wet surface with $r_c = 0\ \mathrm{s\,m^{-1}}$: $E = 0.119\ \mathrm{g\,m^{-2}\,s^{-1}}$.

10.7 Using the Penman-Monteith equation, explain why evaporation can take place even without any available energy (that is, $R_n - G = 0$). Where does the energy that supports the evaporation come from? When this occurs, is the surface air layer statically stable or unstable?

It is clear from the Penman-Monteith equation that E can be positive even if available energy $R_n - G$ vanishes so long as D is positive. That is saying that the saturation vapor pressure deficit drives evaporation. When this happens, there is an exact balance between sensible heat gain and latent heat loss, which is also the wet bulb thermometer scenario.

As the sensible heat flux goes from the atmosphere to the surface, the surface air is statically stable.

10.8 (a) Determine the canopy resistance r_c of a well-watered soybean crop by inverting the Penman-Monteith equation. The latent heat flux λE is 365.0 $\mathrm{W\,m^{-2}}$, the available energy flux $R_n - G$ is 388.8 $\mathrm{W\,m^{-2}}$, the vapor pressure deficit D is 26.6 hPa, the air temperature T_a is 30.1°C, and the aerodynamic resistance r_a is 92 $\mathrm{s\,m^{-1}}$. (b) Repeat the calculation for a temperate evergreen forest under drought stress ($\lambda E = 119.8\ \mathrm{W\,m^{-2}}$, $R_n - G = 487.9\ \mathrm{W\,m^{-2}}$, $D = 7.6$ hPa, $T_a = 17.4$°C, and $r_a = 13\ \mathrm{s\,m^{-1}}$).

The inverted Penman-Monteith equation in favor of the canopy resistance is:

$$r_c = \frac{r_a}{\gamma}\left[\frac{\Delta(R_n - G) + \rho_d c_p D/r_a}{\lambda E} - \Delta\right] - r_a$$

All the parameters are given except for the slope of the saturation vapor pressure curve, which can be obtained by differentiating Original Equation 3.85:

$$\Delta = 6.1365 \exp\left(\frac{17.502t}{240.97 + t}\right)\left[\frac{17.503}{240.97 + t} - \frac{17.502t}{(240.97 + t)^2}\right] \tag{10.1}$$

where Δ is in $\mathrm{hPa\,K^{-1}}$ and t is in °C. At $t = 30.1$°C, $\Delta = 2.46\ \mathrm{hPa\,K^{-1}}$. Plugging in the parameter values given for other variables, we obtain $r_c = 63\ \mathrm{s\,m^{-1}}$ for the soybean crop.

We then repeat the calculation for the forest, noting $\Delta = 1.26\ \mathrm{hPa\,K^{-1}}$ at $t = 17.4$°C. We obtain $r_c = 179\ \mathrm{s\,m^{-1}}$.

10.9 Calculate the reference evaporation rate using a wind speed of 3.5 $\mathrm{m\,s^{-1}}$ at the height of 2.0 m, an air temperature of 17.2°C, a relative humidity of 43.2% and an available energy flux ($R_n - G$) of 201 $\mathrm{W\,m^{-2}}$.

Reference evaporation is calculated using the aerodynamic resistance calculated using Original Equation 10.21:

$$r_a = \frac{1}{k^2 u}\ln\frac{z - d}{z_0}\ln\frac{z - d}{z_{0,h}}$$

We can solve:

$$r_a = \frac{1}{0.4^2 \times 3.5\ \mathrm{m\,s^{-1}}} \ln\frac{2.0\ \mathrm{m} - 0.08\ \mathrm{m}}{0.015\ \mathrm{m}} \ln\frac{2.0\ \mathrm{m} - 0.08\ \mathrm{m}}{0.0015\ \mathrm{m}} = 62\ \mathrm{s\,m^{-1}}$$

The saturation vapor pressure can be calculated using Original Equation 3.85:

$$e_v^* = 6.1365 \exp\left(\frac{17.502t}{240.97 + t}\right) = 19.69\ \mathrm{hPa}$$

Vapor pressure deficit can be calculated using relative humidity:

$$D = [1 - \mathrm{RH}/100] \times e_v^*(T_a) = 11.18\ \mathrm{hPa} \tag{10.2}$$

The saturation slope is given by Equation 10.1 $\Delta = 1.25\ \mathrm{hPa\,K^{-1}}$. Now we can simply use the Penman-Monteith equation to find the reference evaporation rate:

$$E = \frac{1}{2466} \frac{1.25 \times 201 + 1200 \times 11.18/62}{1.25 + 0.66(62 + 70)/62} = 0.071\ \mathrm{g\,m^{-2}\,s^{-1}}.$$

10.10* Using the Priestley-Taylor Equation 10.20 and the surface energy balance Equation 10.13, determine the Bowen ratio for a range of air temperature values (1 to 20°C). Present your result in a graph. According to your result, how should the Bowen ratio of lake systems vary with latitude? If two lakes have the same temperature but are located at different altitudes (one on the Tibetan Plateau and the other at the sea level), which lake is expected to have higher Bowen ratio?

According to the Priestley-Taylor equation for potential evaporation:

$$\lambda E = \alpha \frac{\Delta}{\Delta + \gamma}(R_n - G)$$

and the surface energy balance equation:

$$R_n - G = H + \lambda E$$

we can solve for Bowen ratio, which is defined as the ratio of the surface sensible flux to the surface latent heat flux:

$$\beta = \frac{\Delta + \gamma}{\alpha\Delta} - 1$$

As $\alpha \simeq 1.26$, the expression can be simplified as:

$$\beta = \frac{\gamma}{1.26\Delta} - 0.21$$

Here Δ is the slope of saturation vapor pressure curve given by Equation 10.1. Meanwhile, γ, the psychrometric constant, is 0.66 $\mathrm{hPa\,K^{-1}}$ at sea level. The result can be visualized as the plot below.

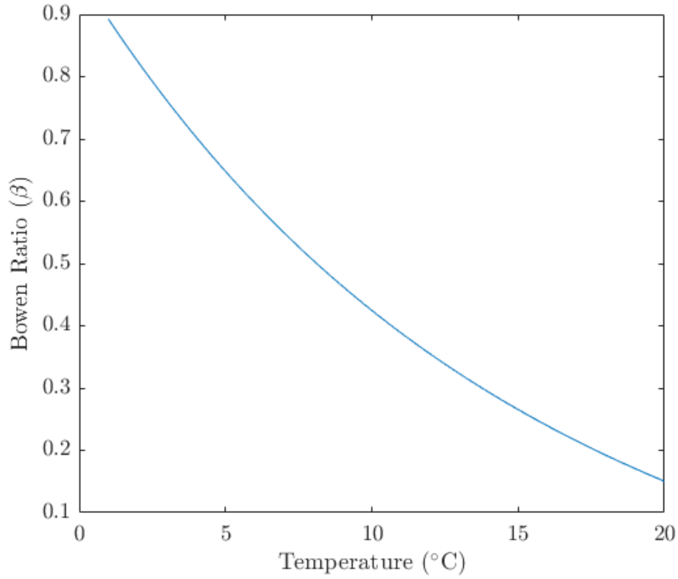


Figure 10.1: Relationship between Lake Bowen Ratio and temperature.

At higher altitude, the temperature is generally lower. Thus, a lake system at higher altitude has higher Bowen ratio. However, if the temperature is constant, the lake at higher altitude would have a lower air pressure, and lower Bowen ratio due to a reduced γ, which is proportional to pressure.

10.11 (a) Confirm that the following expression is the big-leaf model solution for the surface temperature

$$T_s = T_a + \frac{r_a + r_c}{\rho_d c_p} \cdot \frac{\gamma(R_n - G) - \rho_d c_p D/(r_a + r_c)}{\Delta + \gamma(r_a + r_c)/r_a}. \tag{10.3}$$

(b) Green oases are irrigated farmlands whose surface temperature is lower than that of the surrounding dry landscape. Explain why an inversion typically prevails in the surface layer over a green oasis. (c) It is hypothesized that by using white and highly reflective materials as building roofs, a city will turn into a cold island whereby the urban land is cooler than the surrounding rural land. Do you expect unstable lapse conditions or stable inversion over this "white oasis". Why?

(a) The surface temperature is given as:

$$T_s = T_a + \frac{r_a}{\rho_d\, c_p} \cdot H \tag{10.4}$$

where H can be substituted as:

$$H = R_n - G - \lambda E$$

and according to Original Equation 10.17:

$$\lambda E = \frac{\Delta(R_n - G) + \rho_d\, c_p\, D/r_a}{\Delta + \gamma(r_a + r_c)/r_a}$$

Substituting λE and H in Equation 10.4, we get an expression for T_s:

$$T_s = T_a + \frac{r_a}{\rho_d\, c_p} \cdot \left[R_n - G - \frac{\Delta(R_n - G) + \rho_d\, c_p\, D/r_a}{\Delta + \gamma(r_a + r_c)/r_a}\right]$$

Rearranging this equation, we obtain:

$$T_s = T_a + \frac{r_a + r_c}{\rho_d\, c_p} \cdot \frac{\gamma(R_n - G) - \rho_d\, c_p\, D/(r_a + r_c)}{\Delta + \gamma(r_a + r_c)/r_a}$$

which is identical to Original Equation 10.47.

(b) Whether the surface temperature is higher or lower than the air temperature depends on the relative magnitudes of the available energy term and the vapor pressure deficit term. In the case of a green oasis, the hot air is above a moist surface, creating a large vapor pressure deficit. In addition, irrigation reduces the surface resistance to moisture flux. Together, the vapor pressure deficit term can exceed the available energy term in Original Equation 10.47. The result is that the second term on the right-hand side of this equation is negative, and therefore the surface temperature is lower than the air temperature, which is an inversion situation.

(c) Using white and highly reflective materials in urban areas increases the surface albedo, which reduces the available energy term. However, since the surface resistance is very high in urban areas, the deficit term remains smaller than the available energy term. Overall, this leads to unstable lapse conditions, with surface temperature higher than air temperature.

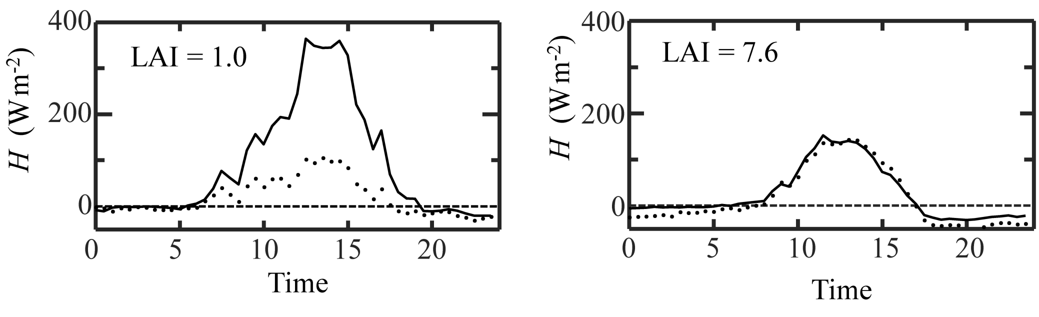


Figure 10.2: (Original Figure 10.13) Comparison of observed (dots) and calculated sensible heat flux (lines) for a soybean field. The calculation is made with Original Equation 10.14 using the surface temperature measured with an infrared thermometer. Data source: Lee et al. (2009).

10.12 Figure 10.2 is a comparison of the sensible heat flux calculated with Original Equation 10.14 against that measured with eddy covariance over a soybean crop. Explains why the calculated flux is biased high at midday in the early growing season (leaf area index 1.0) and why the bias diminishes as the canopy becomes fully closed in the later part of the growing season (leaf area index 7.6).

The sensible heat flux, H, calculated using Original Equation 10.14 is biased too high when the aerodynamic surface temperature T_o (the air temperature at the thermal roughness height $z_{o,h}$) and the radiometric surface temperature T_s (radiative skin temperature) differ. This happens in unstable conditions for bare soil or sparse canopy and T_s is a few degrees higher than T_o, which increases the numerator of Original Equation 10.14, and thus the calculated value of H. A low leaf area index in the growing season represents a sparse canopy and midday conditions are unstable.

This overestimation can be fixed either by using T_o instead of T_s in Original Equation 10.14 or by adding a radiometric resistance term r_m, which represents the diffusion resistance between $z_{o,h}$ and the surface, to the denominator. In the case of a dense canopy, i. e. when the canopy becomes fully closed in the later part of the growing season, T_s and T_o are similar and the calculated H is much closer to the observed H.

10.13 The incoming longwave radiation flux is 437.3 $\mathrm{W\,m^{-2}}$ and the outgoing longwave radiation flux is 494.5 $\mathrm{W\,m^{-2}}$. Calculate the surface temperature assuming (1) that the surface is a blackbody, and (2) that the surface has an emissivity value of 0.97.

Assuming that the surface is a blackbody, the surface temperature can be determined by inverting Stefan- Boltzmann law (Original Equation 10.22):

$$T_s = (L_\uparrow/\sigma)^{1/4} = [494.5/(5.67 \times 10^{-8})]^{1/4} = 305.59 \text{ K}$$

If the surface is not a blackbody, part of the outgoing longwave radiation flux will be due to the reflected incoming longwave radiation. Thus, the equation for surface temperature is modified to:

$$T_s = [(L_\uparrow - L_\downarrow(1-\epsilon))/\epsilon\sigma]^{1/4}$$

where ϵ is the emissivity. Plugging in the values of $L_\uparrow$, $L_\downarrow$, and ϵ, we get:

$$T_s = [(494.5 - 437.3(1-0.97))/(0.97 \times 5.67 \times 10^{-8})]^{1/4} = 305.87 \text{ K}$$

10.14 (a) Show that the local climate sensitivity λ_0 has dimensions of $\mathrm{K\,W^{-1}\,m^2}$. (b) Determine its value for a temperature of 273 K and 293 K.

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a) The local climate sensitivity is given by:

$$\lambda_0 = 1/(4\sigma T_s^3)$$

Plugging in the units of each term, we get:

$$\lambda_0 = 1/(\mathrm{W\,m^{-2}\ K^{-4}\ K^3}) = 1/(\mathrm{W\ m^{-2}\ K^{-1}}) = \mathrm{K\ W^{-1}\ m^2}$$

b) The local climate sensitivity at 273 K is:

$$\lambda_0 = 1/(4 \times 5.67 \times 10^{-8} \times 273^3) = 0.22\ \mathrm{K\ W^{-1}\ m^2}$$

For 293 K, it is:

$$\lambda_0 = 1/(4 \times 5.67 \times 10^{-8} \times 293^3) = 0.17\ \mathrm{K\ W^{-1}\ m^2}$$

10.15 A typical energy redistribution factor is 6 at midday and 2 at midnight. Estimate the anthropogenic heat contribution to the urban heat island if the anthropogenic heat flux is 40 $\mathrm{W\,m^{-2}}$.

Use local climate sensitivity $\lambda_0 = 0.2$ K m^2 W^{-1}. Use Term 5 of Original Equation 10.46 to get the contribution to the urban heat island from anthropogenic heat release:

$$\Delta T_s \simeq \frac{\lambda_0}{1+f} Q_A$$

At midday:

$$\Delta T_s \simeq \frac{0.2\ \mathrm{K\ m^2\ W^{-1}}}{1+6} \times 40\ \mathrm{W\ m^{-2}} = 1.14\ \mathrm{K}$$

At midnight:

$$\Delta T_s \simeq \frac{0.2\ \mathrm{K\ m^2\ W^{-1}}}{1+2} \times 40\ \mathrm{W\ m^{-2}} = 2.67\ \mathrm{K}$$

Table 10.2: (Original Table 10.1) Diagnostic variables of the surface energy balance at midday in the summer for urban and rural land in Eastern United States: $K_\downarrow$, incoming solar radiation; $L_\downarrow$, incoming longwave radiation; α, surface albedo; r_T, total heat resistance; β, Bowen ratio; Q_S, heat storage; Q_A, anthropogenic heat flux; T_a, air temperature at the blending height.

Surface	$K_\downarrow$ $\mathrm{W\,m^{-2}}$	$L_\downarrow$ $\mathrm{W\,m^{-2}}$	α	r_T $\mathrm{s\,m^{-1}}$	β	Q_S $\mathrm{W\,m^{-2}}$	Q_A $\mathrm{W\,m^{-2}}$	T_a K
Urban	709	418	0.18	62	2.3	125	57	299.3
Rural	709	418	0.11	35	1.7	88	0	299.3

10.16* Using the diagnostic data in Table 10.2, estimate the contributions to the urban heat island intensity from changes in surface albedo, convection efficiency, Bowen ratio (or evaporation) and heat storage and from anthropogenic heat release.

Use Original Equation 10.46

$$\begin{aligned}\Delta T_s \quad &\simeq \quad \frac{\lambda_0}{1+f}(\Delta S) + \frac{\lambda_0}{(1+f)^2}(R_n^* - Q_S + Q_A)(\Delta f_1) \\ &+\frac{\lambda_0}{(1+f)^2}(R_n^* - Q_S + Q_A)(\Delta f_2) \\ &+\frac{\lambda_0}{1+f}(\Delta Q_S) + \frac{\lambda_0}{1+f}Q_A\end{aligned}$$

The first term on the right side of Original Equation 10.46 represents contributions to the urban heat island from changes in albedo, where $\lambda_0 = 0.1645\ \mathrm{K\,m^2\ W^{-1}}$ and

$$\begin{aligned}f &= \frac{\rho_d c_p \lambda_0}{r_T}(1+\frac{1}{\beta}) \\ &= \frac{1.225\ \mathrm{kg\,m^{-3}} \times 1004\ \mathrm{J\,kg^{-1}\,K^{-1}} \times 0.1645\ \mathrm{K\,m^2\,W^{-1}}}{62\ \mathrm{s\,m^{-1}}} \times (1+\frac{1}{2.3}) \\ &= 4.68\end{aligned}$$

The contribution from changes in albedo is therefore:

$$\begin{aligned}\frac{\lambda_0}{1+f}(\Delta S) &= \frac{\lambda_0}{1+f}(\Delta((1-\alpha))K_\downarrow \\ &= \frac{0.1645\ \mathrm{K\,m^2\,W^{-1}}}{1+4.68}((1-0.18)-(1-0.11)) \times 709\ \mathrm{W\,m^{-2}} \\ &= -1.44\ \mathrm{K}.\end{aligned}$$

The contribution from changes in convection efficiency is represented in the second term on the right side of Original Equation 10.46, where

$$\begin{aligned}R_n^* &= (1-\alpha)K_\downarrow + L_\downarrow - \sigma T_a^4 \\ &= (1-0.18) \times 709\ \mathrm{W\,m^{-2}} + 418\ \mathrm{W\,m^{-2}} - 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\ K^{-4}} \times (299.3\ \mathrm{K})^4 \\ &= 544.38\ \mathrm{W\,m^{-2}}\end{aligned}$$

and

$$\Delta f_1 = f\frac{\Delta r_T}{r_T} = 4.68\frac{62\ \mathrm{s\,m^{-1}} 35\ \mathrm{s\,m^{-1}}}{62\ \mathrm{s\,m^{-1}}} = 2.04$$

Therefore, the contribution from changes in convection efficiency is:

$$\begin{aligned}&\frac{\lambda_0}{(1+f)^2}(R_n^* - Q_S + Q_A)(\Delta f_1) \\ &= \frac{0.1645\ \mathrm{K\,m^2\ W^{-1}}}{(1+4.68)^2}(544.38\ \mathrm{W\,m^{-2}} - 125\ \mathrm{W\,m^{-2}} + 57\ \mathrm{W\,m^{-2}})(2.04) \\ &= 4.96\ \mathrm{K}\end{aligned}$$

The contribution from changes in the Bowen ratio is represented in the third term on the right side of Original Equation 10.46, where:

$$\begin{aligned}
\Delta f_2 &= \frac{\rho_d c_p \lambda_0}{r_T}\left(\frac{\Delta\beta}{\beta^2}\right) \\
&= \frac{1.225\text{kg}\,\text{m}^{-3} \times 1004\ \text{J}\,\text{kg}^{-1}\,\text{K}^{-1} \times 0.1645\ \text{K}\,\text{m}^2\,\text{W}^{-1}}{62\ \text{s}\,\text{m}^{-2}} \times \frac{2.3 - 1.7}{2.3^2} \\
&= 0.37
\end{aligned}$$

So the contribution from changes in the Bowen ratio is:

$$\begin{aligned}
&\frac{\lambda_0}{(1+f)^2}(R_n^* - Q_S + Q_A)(\Delta f_2) \\
&= \frac{0.1645\ \text{K}\,\text{m}^2\,\text{W}^{-1}}{(1+4.68)^2}(544.38\ \text{W}\,\text{m}^{-2} - 125\ \text{W}\,\text{m}^{-2} + 57\ \text{W}\,\text{m}^{-2})(0.37) \\
&= 0.90\ \text{K}
\end{aligned}$$

The contribution from changes in heat storage is represented by the forth term on the right side of Original Equation 10.46:

$$\begin{aligned}
&\frac{-\lambda_0}{1+f}(\Delta Q_S) \\
&= \frac{-0.1645\ \text{K}\,\text{m}^2\,\text{W}^{-1}}{1+4.68}(88\ \text{W}\,\text{m}^{-2} - 125\ \text{W}\,\text{m}^{-2}) \\
&= 1.07\ \text{K}
\end{aligned}$$

The contribution from changes in anthropogenic heat release is represented by the fifth term on the right side of Original Equation 10.46:

$$\begin{aligned}
&\frac{\lambda_0}{1+f}Q_A \\
&= \frac{0.1645\ \text{K}\,\text{m}^2\,\text{W}^{-1}}{1+4.68}(57\ \text{W}\,\text{m}^{-2}) \\
&= 1.65\ \text{K}
\end{aligned}$$

In the above calculation, the urban variables are used to define the baseline state. The result can be improved by repeating the calculation using the rural variables as the baseline, and average the results from the two sets of calculation.

10.17* In response to high human mortality in a heat wave event in 1995, the City of Chicago has been promoting use of reflective roof materials for urban heat mitigation. According to a satellite study, this practice increased the citywide albedo by about 0.02 from 1995 to 2010. Estimate the surface temperature reduction caused by the albedo change. (Hint: Use the data provided in Original Table 10.1 for your calculation.)

We first calculate the local climate sensitivity from Original Equation 10.36, with the Stefan-Boltzmann constant σ and T_a from Table 10.1:

$$\begin{aligned}\lambda_0 &= 1/(4\sigma T_s^3) \simeq 1/(4\sigma T_a^3)\\ &= 1/[(4) \times (5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}) \times (299.3 \text{ K})^3]\\ &= 0.164 \text{ K m}^2 \text{ W}^{-1}\end{aligned}$$

Next we use Original Equation 10.41 to get the energy redistribution factor, using values from Table 10.1:

$$\begin{aligned}f &= \frac{\rho_d c_p \lambda_0}{r_T}(1+\frac{1}{\beta})\\ &= \frac{1.225 \text{ kg m}^{-3} \times 1004 \text{ J kg}^{-1} \text{ K}^{-1} \times 0.164 \text{ K m}^2 \text{ W}^{-1}}{62 \text{ s m}^{-1}}(1+\frac{1}{2.3})\\ &= 4.68\end{aligned}$$

Then we use Original Equation 10.33 to get change in the net shortwave radiation flux, using values from Table 10.1 for $K_\downarrow$:

$$\begin{aligned}S &= (1-\alpha)K_\downarrow\\ \Delta S &= -\Delta\alpha K_\downarrow\\ &= -0.02 \times 709 \text{ W m}^{-2}\\ &= -14.18 \text{ W m}^{-2}\end{aligned}$$

Finally we use Term 1 of Original Equation 10.46 to get the temperature change caused by changes in albedo:

$$\begin{aligned}\Delta T_s &\simeq \frac{\lambda_0}{1+f}(\Delta S)\\ &= \frac{0.164 \text{ K m}^2 \text{ W}^{-1}}{1+4.68}(-14.18 \text{ W m}^{-2})\\ &= -0.41 \text{ K}\end{aligned}$$

10.18 Using the two-source model, compute the soil evaporation and the plant transpiration of an ecosystem with leaf area index of 4. The canopy resistance r_c is 50 s m^{-1} and the ground resistance r_g is 500 s m^{-1}. The ground heat flux is approximated as $G = 0.2R_{n,g}$. The meteorological conditions are: aerodynamic resistance $r_a = 42$ s m^{-1}, net radiation $R_n = 400$ W m^{-2}, vapor pressure deficit $D = 20$ hPa, and air temperature $T_a = 25°\text{C}$.

Use the empirical light extinction coefficient $a = 0.7$, the psychrometric constant $\gamma = 0.66$ hPa K^{-1}, the latent heat of vaporization $\lambda = 2466$ J g^{-1}, and the slope of saturation vapor pressure $\Delta = 1.90$ hPa K^{-1}.

The net radiation of the canopy layer is obtained from Original Equation 10.25:

$$\begin{aligned} R_{n,c} &= R_n[1 - \exp(-aL)] \\ &= (400\ \mathrm{W\,m^{-2}})[1 - \exp(-0.7 \times 4)] \\ &= 375.68\ \mathrm{W\,m^{-2}} \end{aligned}$$

The net radiation of the ground surface is obtained from Original Equation 10.26:

$$\begin{aligned} R_{n,g} &= R_n \exp(-aL) \\ &= (400\ \mathrm{W\,m^{-2}})\exp(-0.7 \times 4)] \\ &= 24.32\ \mathrm{W\,m^{-2}} \end{aligned}$$

Canopy transpiration is obtained from Original Equation 10.27:

$$\begin{aligned} E_c &= \frac{1}{\lambda}\frac{\Delta R_{n,c} + \rho_d c_p D/r_a}{\Delta + \gamma(r_a + r_c)/r_a} \\ &= \frac{1}{2466\ \mathrm{J\,g^{-1}}} \times \\ &\frac{(1.90\,\mathrm{hPa\,K^{-1}})(375.68\,\mathrm{W\,m^{-2}}) + (1.225\,\mathrm{kg\,m^{-3}})(1004\,\mathrm{J\,kg^{-1}\,K^{-1}})(20\,\mathrm{hPa})/(42\,\mathrm{s\,m^{-1}})}{(1.90\,\mathrm{hPa\,K^{-1}}) + (0.66\,\mathrm{hPa\,K^{-1}})(42\ \mathrm{s\,m^{-1}} + 50\,\mathrm{s\,m^{-1}})/(42\ \mathrm{s\,m^{-1}})} \\ &= 0.157\ \mathrm{g\ m^{-2}\ s^{-1}} \end{aligned}$$

Soil evaporation is obtained from Original Equation 10.28:

$$\begin{aligned} E_g &= \frac{1}{\lambda}\frac{\Delta(R_{n,g} - G) + \rho_d c_p D/r_a}{\Delta + \gamma(r_a + r_g)/r_a} \\ &= \frac{1}{2466}\frac{(1.90)(24.32 - (0.2)(24.32)) + (1.225)(1004)(20)/42}{1.90 + (0.66)(42 + 500)/42} \\ &= 0.024\ \mathrm{g\ m^{-2}\ s^{-1}} \end{aligned}$$

Total evaporation flux is obtained from Original Equation 10.29:

$$\begin{aligned} E &= E_c + E_g \\ &= 0.157\ \mathrm{g\ m^{-2}\ s^{-1}} + 0.024\ \mathrm{g\ m^{-2}\ s^{-1}} \\ &= 0.182\ \mathrm{g\ m^{-2}\ s^{-1}} \end{aligned}$$

10.19* Using the two-source model, compute the fraction of total evaporation contributed by plants (E_c/E) as a function of leaf area index L for ecosystems with moist (ground resistance r_g of 500 $\mathrm{s\,m^{-1}}$) and dry soil (r_g of 2000 $\mathrm{s\,m^{-1}}$). The mean stomatal resistance r_s is 200 $\mathrm{s\,m^{-1}}$. The canopy resistance r_c is parameterized by Original Equation 10.18 and the ground heat flux is approximated as $G = 0.2R_{n,g}$. The meteorological conditions are the same as in Problem 10.18.

According to Original Equations 10.25 and 10.26, the net radiation of the canopy layer $R_{n,c}$ and that of ground surface $R_{n,g}$ are given by:

$$R_{n,c} = 400 \times [1 - \exp(-0.7L)]$$
$$R_{n,g} = 400 \times \exp(-0.7L)$$

According to Original Equation 10.18, the canopy resistance $r_c = 200/L$. According to Original Equation 10.27 and 10.28, we have:

$$E_c = \frac{1}{2.466 \times 10^6} \frac{190 \times 400 \times [1 - \exp(-0.7L)] + 1.20 \times 1004 \times 2000/42}{190 + 66 \times (42 + 200/L)/42}$$
$$E_g = \frac{1}{2.466 \times 10^6} \frac{190 \times 320 \times \exp(-0.7L) + 1.20 \times 1004 \times 2000/42}{190 + 66 \times (42 + r_g)/42}$$

The ratio, $E_c/(E_c + E_g)$ or E_c/E, is presented in Figure 10.3 as a function of L for two soil moisture conditions.

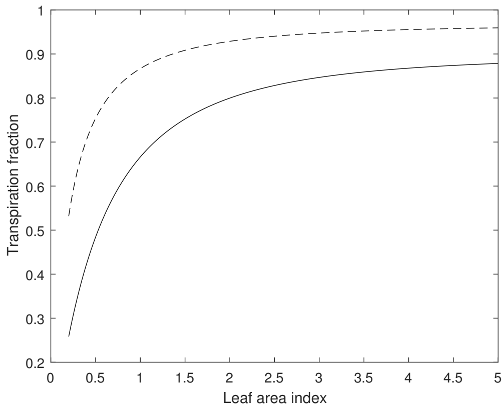

Figure 10.3: Fractional contribution of plant transpiration to total evapotranspiration as a function of leaf area index: black solid line, wet surface; black dashed line, dry surface.

10.20* Atmospheric modelers do not use equations of the Penman-Monteith type to compute the surface flux boundary conditions because the surface net radiation R_n is a predicted variable, not a forcing variable. Instead, they must first solve the surface temperature T_s and then use the resistance formulae to compute the fluxes. Derive an analytical expression for T_s using Equations 10.14, 10.15 and 10.37. Assume in your derivation that the ground heat flux is negligible.

We rearrange the surface energy balance equation:

$$(1 - \alpha)\, K_\downarrow + L_\downarrow - \sigma T_s^4 = H + LE$$

to

$$(1-\alpha)\,K_{\downarrow} + L_{\downarrow} - \sigma T_a^4 + 4\sigma T_a^3\,(T_s - T_a) = H + LE$$

Using the resistance formulation and the definition of apparent net radiation:

$$R_n^* = (1-\alpha)\,K_{\downarrow} + L_{\downarrow} - \sigma T_a^4$$

we obtain:

$$R_n^* + 4\sigma T_a^3\,(T_s - T_a) = \rho_d c_p \frac{T_s - T_a}{r_a} + \lambda \rho_d \frac{s_v^* - s_v}{r_a + r_c}$$

Using the linear expression for s_v^* (Original Equation 10.16), the above equation becomes:

$$R_n^* + 4\sigma T_a^3\,(T_s - T_a) = \rho_d c_p \frac{T_s - T_a}{r_a} + \lambda \rho_d \left[\frac{0.621}{p_d} D + \frac{0.621}{p_d} \Delta(T_s - T_a)\right] \frac{1}{r_a + r_c}$$

This equation can be rearranged to obtain a solution of $(T_s - T_a)$.

Chapter 11

Budgets of Heat, Water Vapor, and Trace Gases in the Atmospheric Boundary Layer

11.1 Under what conditions is the entrainment velocity equal to the time rate of change of the boundary layer height? Assuming that these conditions are satisfied, estimate the entrainment velocity using the profile data shown in Original Figure 6.11.

According to the prognostic equation for boundary layer height (Original Equation 11.21):

$$\frac{\partial z_i}{\partial t} = \overline{w} - \frac{(\overline{w'\theta'})_{z_i}}{\Delta_\theta} = \overline{w} - w_e,$$

we see that the entrainment velocity is equal to the time rate of change of the boundary layer height when the vertical velocity is zero. When there is no large scale flow divergence or convergence, this condition will be satisfied.

Assuming that the condition above is satisfied, according to Original Figure 6.11, the estimated ABL height is shown in the following table:

time	09:15	11:15	13:15	15:15	17:15
ABL height (m)	400	1400	1500	1700	1750

The entrainment velocity is the same as ABL growth rate:

time	09:15	11:15	13:15	15:15	17:15
entrainment velocity (m s^{-1})	-	0.14	0.014	0.028	0.007

11.2 The entrainment ozone flux in a marine boundary layer topped by stratocumulus clouds is -8.1 ppb cm s^{-1}, and the ozone concentration jump across the capping inversion is approximately 15 ppb (Faloona et al., 2005). What is the entrainment velocity?

By modifying Original Equation 11.46, we have:

$$w_e = -\frac{(\overline{w's'_o})_{z_i}}{\Delta_o} = -\frac{-8.1 \text{ ppb cm s}^{-1}}{15 \text{ ppb}} = 0.54 \text{ cm s}^{-1}$$

where $(\overline{w's'_o})_{z_i}$ is the entrainment ozone flux and Δ_o is the ozone concentration jump.

11.3 For a heat flux of -0.03 K m s^{-1} below the capping inversion, an inversion strength of 1.7 K, a horizontal flow divergence rate of 2×10^{-6} s^{-1}, and a boundary layer height of 650 m, find the rate of boundary layer growth.

By using Original Equations 11.21 and 3.18, we have:

$$\begin{aligned}\frac{\partial z_i}{\partial t} &= \overline{w} - \frac{(\overline{w'\theta'})_{z_i}}{\Delta_\theta} = \int_0^{z_i} -\left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{y}}{\partial y}\right) \mathrm{d}z' - \frac{(\overline{w'\theta'})_{z_i}}{\Delta_\theta} \\ &= -2 \times 10^{-6} \text{ s}^{-1} \times 650 \text{ m} - \frac{-0.03 \text{ K m s}^{-1}}{1.7 \text{ K}} = 0.016 \text{ m s}^{-1}\end{aligned}$$

11.4 The atmospheric boundary layer is in a quasi steady state. Estimate the mean vertical velocity in cumulus clouds $\overline{w}_c$ if the cumulus cloud fraction is 0.04, the entrainment velocity is 0.02 m s^{-1} and there is no large-scale subsidence motion.

At a quasi-steady state, the mean subsidence of air outside the clouds $\overline{w}_f$ is balanced by the entrainment flux. Therefore, we have:

$$\overline{w}_f = \frac{(\overline{w'\theta'}_{z_i})}{\Delta_\theta} = -w_e$$

By combining the above equation with the mass conservation equation:

$$\sigma_c \overline{w}_c + (1 - \sigma_c)\overline{w}_f = 0$$

we have the mean vertical velocity in the clouds as:

$$\overline{w}_c = \frac{1-\sigma_c}{\sigma_c} w_e = \frac{1-0.04}{0.04} \times 0.02 \text{ m s}^{-1} = 0.48 \text{ m s}^{-1}$$

11.5 A reasonable estimate of the mean vertical velocity $\overline{w}_c$ in shallow cumulus clouds formed in fair weather conditions can be obtained from the convective velocity scale w_*, as

$$\overline{w}_c \simeq w_* = \left[\frac{g}{\overline{\theta}_s}(\overline{w'\theta'})_s z_i\right]^{1/3}, \tag{11.1}$$

(Vilá-Guerau de Arellano et al., 2016). The surface heat flux $(\overline{w'\theta'})_s$ is 0.31 K m s^{-1}, the entrainment flux is given by the closure Equation 11.36 (with $A_T = 0.2$), the inversion temperature jump Δ_θ is 1.5 K, the boundary layer height z_i is 950 m, the large-scale subsidence is negligible, and the boundary layer is in a quasi steady state. Determine the fraction of the sky occupied by clouds.

According to Original Equation 11.36 (with $A_T = 0.2$), the entrainment heat flux is given by:

$$\begin{aligned}(\overline{w'\theta'})_{z_i} &= -A_T\,(\overline{w'\theta'})_s \\ &= -0.2 \times 0.31\,\mathrm{K\,m\,s^{-1}} \\ &= -0.062\,\mathrm{K\,m\,s^{-1}}\end{aligned}$$

In a quasi-steady state, the subsidence velocity caused by cloud venting is given by:

$$\begin{aligned}\overline{w}_f &= \frac{(\overline{w'\theta'})_{z_i}}{\Delta_\theta} = -w_e \\ &= \frac{-0.062\,\mathrm{K\,m\,s^{-1}}}{1.5\,\mathrm{K}} = -0.041\,\mathrm{m\,s^{-1}}\end{aligned}$$

Assuming $\overline{\theta}_s = 287$ K, w_c can be calculated with:

$$\begin{aligned}\overline{w}_c \simeq w_* &= \left[\frac{g}{\theta_s}(\overline{w'\theta'})_s\, z_i\right]^{1/3} \\ &= \left[\frac{9.81\,\mathrm{m\,s^{-2}}}{287\,\mathrm{K}} \times 0.31\,\mathrm{K\,m\,s^{-1}} \times 950\,\mathrm{m}\right]^{1/3} \\ &= 2.16\,\mathrm{m\,s^{-1}}\end{aligned}$$

According to Original Equation 11.24:

$$\sigma_c\,\overline{w}_c + (1 - \sigma_c)\,\overline{w}_f = 0$$

The fraction of the sky occupied by clouds is:

$$\sigma_c = \frac{\overline{w}_f}{(\overline{w}_f - \overline{w}_c)} = 0.19$$

11.6 Air temperature in the free atmosphere is generally lower than that in the boundary layer. Why do we say that entrainment of the free atmospheric air into the boundary layer will cause warming of the boundary layer?

Although air temperature in the free atmosphere is generally lower than that in the boundary layer, potential temperature is higher in the free atmosphere. Potential temperature is the temperature an air parcel would have if it were moved adiabatically to the sea level. Therefore, when the free atmospheric air is entrained into the boundary layer, the air in the boundary layer will become warmer.

11.7 The surface heat flux is given by

$$(\overline{w'\theta'})_s(t) = 0.1\sin(\pi t/12) \qquad (0 < t < 12) \tag{11.2}$$

where $(\overline{w'\theta'})_s$ is in K m s^{-1}, and t is time in hours since sunrise. The boundary layer height is 0 m at sunrise. The large-scale flow divergence rate is zero. The potential temperature gradient in the free atmosphere γ_θ is 3.3 K km^{-1}. Estimate the boundary layer height and the inversion temperature jump at hourly intervals from $t = 1$ h to 10 h.

Employing Original Equations 11.37 and 11.38, we have:

$$\begin{aligned} z_i &= \left[\frac{2.8}{\gamma_\theta}\int_0^t (\overline{w'\theta'})_s dt'\right]^{1/2} \\ &= \left[\frac{0.28}{\gamma_\theta}\int_0^t \sin\left(\frac{\pi t'}{12}\right)dt'\right]^{1/2} \\ &= \left[\frac{0.28}{\gamma_\theta}\frac{12\times 3600}{\pi}\left(1-\cos\left(\frac{\pi t}{12}\right)\right)\right]^{1/2}. \end{aligned}$$

Note that $\overline{w'\theta'}$ is in unit of K m s^{-1} while the integration is performed in hours, which gives rise to a 3600 s h^{-1} conversion parameter in the integration. Accordingly:

$$\Delta_\theta = 0.14\gamma_\theta z_i.$$

The estimated values are shown in Table 11.1

Table 11.1: Boundary-layer height and the inversion temperature jump for Problem 11.7.

time (hour)	z_i (m)	$\Delta_\theta(K)$
1	199.39	0.09
2	395.37	0.18
3	584.58	0.27
4	763.79	0.35
5	929.93	0.43
6	1080.16	0.50
7	1211.91	0.56
8	1322.92	0.61
9	1411.30	0.65
10	1475.53	0.68

11.8 Repeat the calculation of Problem 11.7. But this time the boundary layer height z_i and the temperature inversion jump Δ_θ are 200 m and 2.3 K at sunrise and the product $z_i\Delta_\theta$ is invariant with time.

As the product $z_i \Delta_\theta$ is invariant with time, we should use Original Equation 11.39. Integrating this equation, we get:

$$\begin{aligned} z_i &= \left[z_i(0)^2 + \frac{0.2}{\gamma_\theta} \frac{12 \times 3600}{\pi} \left(1 - \cos\left(\frac{\pi t}{12} \right) \right) \right]^{1/2} \\ &= \left[(200\text{m})^2 + \frac{0.2}{\gamma_\theta} \frac{12 \times 3600}{\pi} \left(1 - \cos\left(\frac{\pi t}{12} \right) \right) \right]^{1/2}. \end{aligned}$$

The inversion temperature jump is given by:

$$\Delta_\theta = \frac{200 \text{ m} \times 2.3 \text{ K}}{z_i}$$

The estimated values are shown in Table 11.2.

Table 11.2: Boundary-layer height and the inversion temperature jump for Problem 11.8.

time (hour)	z_i (m)	$\Delta_\theta (K)$
1	261.53	1.76
2	389.43	1.18
3	533.01	0.86
4	675.79	0.68
5	810.98	0.57
6	934.56	0.49
7	1043.60	0.44
8	1135.82	0.40
9	1209.42	0.38
10	1262.99	0.36

11.9 The surface heat flux is given by Original Equation 11.63. The entrainment heat flux is given by the the entrainment parameterization Original Equation 11.36 with $A_T = 0.2$. The initial column mean potential temperature is 281.1 K. Using the boundary layer height obtained in Problem 11.8, predict the column mean potential temperature at hourly intervals from $t = 1$ h to 10 h.

To predict the evolution of boundary layer temperature $\overline{\theta}_m$, we employ Original Equation 11.32 with $A_T = 0.2$:

$$\begin{aligned} z_i \frac{\partial \overline{\theta}_m}{\partial t} &= (\overline{w'\theta'})_s - (\overline{w'\theta'})_{z_i} \\ &= 1.2 (\overline{w'\theta'})_s \end{aligned}$$

$$\frac{\partial \overline{\theta}_m}{\partial t} = \frac{1.2 \times 0.1 \times \sin\left(\frac{\pi t}{12}\right)}{z_i} \times 3600$$

Note that we include 3600 s h^{-1} to convert time unit from s to h.

The potential temperature at hour i is given by:

$$\overline{\theta}_m = \overline{\theta}_{m,0} + \sum_1^i \left(\frac{\partial \overline{\theta}_m}{\partial t} \right)_i$$

Table 11.3: Potential temperature and its time rate of change for Problem 11.9.

time (hour)	$\frac{\partial\overline{\theta}_m}{\partial t}$ (K h^{-1})	$\overline{\theta}_m$(K)
1	0.43	281.53
2	0.55	282.08
3	0.57	282.66
4	0.55	283.21
5	0.51	283.72
6	0.46	284.19
7	0.40	284.59
8	0.33	284.91
9	0.25	285.17
10	0.17	285.34

The estimated values for $\partial\overline{\theta}_m/\partial t$ and $\overline{\theta}_m$ are shown in Table 11.3.

11.10* The surface heat flux is given by Original Equation 11.63. The initial z_i is 200 m, the initial Δ_θ is 2.3 K, and γ_θ is 3.3 K km^{-1}. Solve numerically the boundary layer height z_i and the inversion strength Δ_θ for $t = 1$ h to 10 h from Equations 11.21, 11.32, 11.35 and 11.36. The large-scale mean vertical velocity $\overline{w}$ is zero. Compare these numerical solutions with the results obtained in Problems 11.7 and 11.8.

Approach 1:

Steps for numerical solution (using a time interval of 1 second):

1. Calculate surface heat flux $(\overline{w'\theta'})_s$ from Original Equation 11.63.

2. Calculate entrainment heat flux $(\overline{w'\theta'}_{z_i})$ using the result of step 1 and an entrainment ratio (A_T) of 0.2 in Original Equation 11.36.

3. Calculate change in boundary layer height (z_i) and mean boundary layer potential temperature $(\overline{\theta}_m)$ during one time step using Original Equations 11.21 and 11.32, respectively.

4. Calculate change in inversion strength (Δ_θ) during one time step using Original Equation 11.35.

5. Find the new z_i, $\overline{\theta}_m$, and Δ_θ for the next time step.

6. Start again from step 1.

Figure 11.1 shows the change in boundary layer height and inversion strength during ten hours after sunrise.

The MATLAB code is below:

```
%time steps
t=[1:1:36000];
%Initial boundary layer height
```

```
z=200;
%Inversion jump
Del=2.3;
%Potential temperature gradient in free atmosphere
lamb=.0033;
%Large-scale mean vertical velocity
w=0;
%Mean potential temperature of boundary layer
Thetam=281.1;
%Change in boundary layer height per time step
delz=0;
%Change in inversion jump per time step
delDel=0;
%Change in potential temperature of boundary layer per time step
delThetam=0;
for i=1:length(t)
    %Original Equation 11.63
    H=0.1*sin(3.14*(t(i)/3600)/12);
    %Original Equation 11.36
    Hz=-0.2*H;
    %Original Equation 11.32
    delThetam=(H-Hz)/z;
    %Original Equation 11.21
    delz=w-(Hz/Del);
    %Original Equation 11.35
    delDel=lamb*(delz-w)-delThetam;
    %Boundary layer height for next time step
    z=z+delz;
    %Inversion jump for next time step
    Del=Del+delDel;
    %Mean potential temperature of boundary layer for next time step
    Thetam=Thetam+delThetam;
    %Create dataset for boundary layer height, mean potential temperature,
    %and inversion jump
    BLM1(i)=z;
    Theta1(i)=Thetam;
    Inv1(i)=Del;
end
%Plot the required variables against time after this.
```

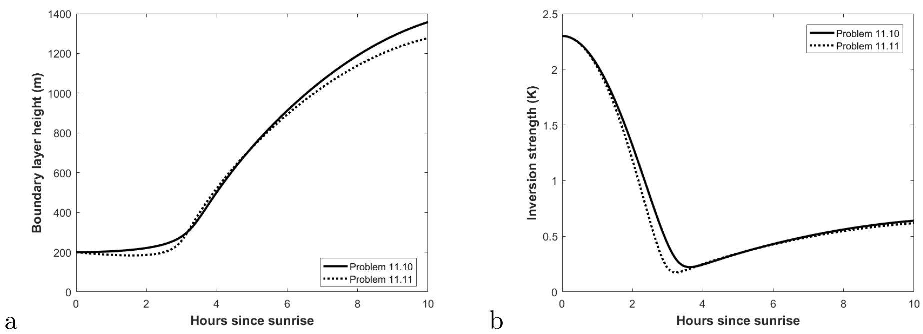

Figure 11.1: Change in boundary layer height (a) and inversion strength (b) during the first ten hours after sunrise for Problem 11.10 and 11.11.

The boundary layer height calculated using the numerical solution is greater than the results of Problems 11.7 and 11.8.

For the inversion jump, the values calculated here are higher than those in problem 11.7, but lower than those calculated in Problem 11.8.

Approach 2:

By combining Original Equations 11.21, 11.32, 11.35, and 11.36, we obtain equations for the time rates of change of z_i and Δ_θ:

$$\frac{\partial z_i}{\partial t} = \frac{0.2(\overline{w'\theta'})_s}{\Delta_\theta} = \frac{0.02\sin(\pi t/43200)}{\Delta_\theta}$$
$$\frac{\partial \Delta_\theta}{\partial t} = (\frac{6.6\times 10^{-5}}{\Delta_\theta} - \frac{0.12}{z_i})\sin(\pi t/43200)$$

The solutions for z_i and Δ_θ are found by integrating these equations with respect to time (Fig. 11.2). The R code for this numerical method is given below.

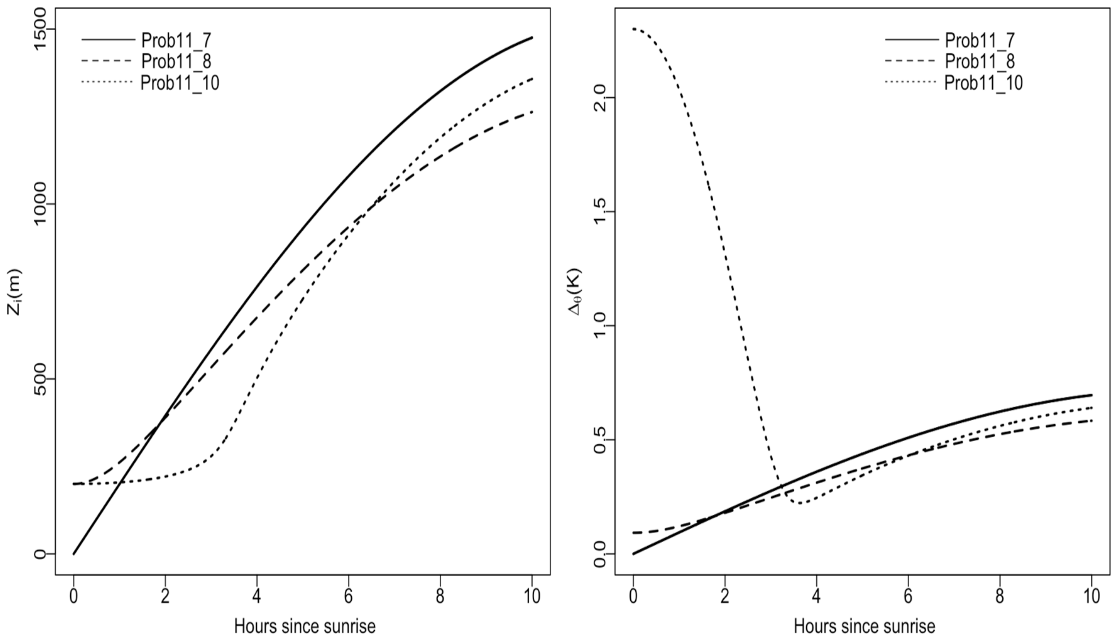

Figure 11.2: Comparison of three solutions for the boundary layer height (left) and the inversion temperature jump (right).

```
# R code
#Prob. 11.10: mean vertical velocity(w) is zero;
#set model parameters
pars <- c(r=3.3/10^3,#gamma  K/km^3
          )
#set initial values
yini <- c(Z = 200, #z_i boundary height m
          D = 2.3  #Δ_theta  inversion strength K
           )# change the Z and D values into 0.001 and 0.001 and then get the ...
               numerical solution of Prob. 11.7
times <- seq(0, 10*3600, by = 60)
#set differential equations
LVmod0D <- function(t, State, Pars){
  with(as.list(c(t,State, Pars)),{
    dZ <- 0.02*sin(pi/12*t/3600)/D#dZ means partial differential equation of z_i
    dD <- 0.1*(0.2*r/D-1.2/Z)*sin(pi/12*t/3600)#dD means partial differential ...
        equation of Δ_theta
```

```
    #dZ<-(-dD*Z/D)# remove pound sign in front of this equation and then get ...
        the numerical solution of Prob. 11.8. This equation means z_i*Δ_theta ...
        keeps constant with time
    return(list(c(dZ, dD)))
  })
}
#out1<-ode(func=LVmod0D,y=yini,parms=pars,times=times)# output values
#out1
```

11.11* Repeat the calculation of Problem 11.10 but with a large-scale mean vertical velocity of -0.5×10^{-2} m s^{-1}. How does large-scale flow subsidence affect the ABL growth and the capping inversion strength?

Approach 1:

The steps are identical to Problem 11.10, except a large-scale mean vertical velocity ($\overline{w}$) of -0.5×10^{-2} m s^{-1} is included in Original Equation 11.21 (step 4).

Figure 11.1 shows the change in boundary layer height and inversion strength during ten hours after sunrise. As seen in Fig. 11.1, the large-scale flow subsidence restricts the maximum boundary layer height and actually reduces the height of the boundary layer from the initial value for the first few hours. Similarly, the inversion strength is reduced by the flow subsidence, especially in the early morning.

Approach 2:

Now that the large scale flow means $\overline{w}$ is not zero, the time rates of z_i and Δ_θ change are given as:

$$\frac{\partial z_i}{\partial t} = -0.5\times 10^{-2} + \frac{0.02\times \sin(\pi t/43200)}{\Delta_\theta}$$

$$\frac{\partial \Delta_\theta}{\partial t} = (\frac{6.6\times 10^{-5}}{\Delta_\theta} - \frac{0.12}{z_i})\sin(\pi t/43200)$$

The resulting z_i is smaller than obtained in Problem 11.10 (Figure 11.3).

11.12 For an entrainment velocity of 0.41 cm s^{-1}, a CO_2 concentration jump across the capping inversion of 20 mg kg^{-1}, and a zero subsidence velocity, find the entrainment CO_2 flux.

Since there is no subsidence, we can use the entrainment similarity formula (Original Equation 11.43):

$$w_e = -\frac{(\overline{w's'_c})_{z_i}}{\Delta_c}$$

$$(\overline{w's'_c})_{z_i} = -w_e\times\Delta_c = -20\times 0.41\times 10^{-3} = -0.0082 \text{ mg kg}^{-1}\text{ m s}^{-1}$$

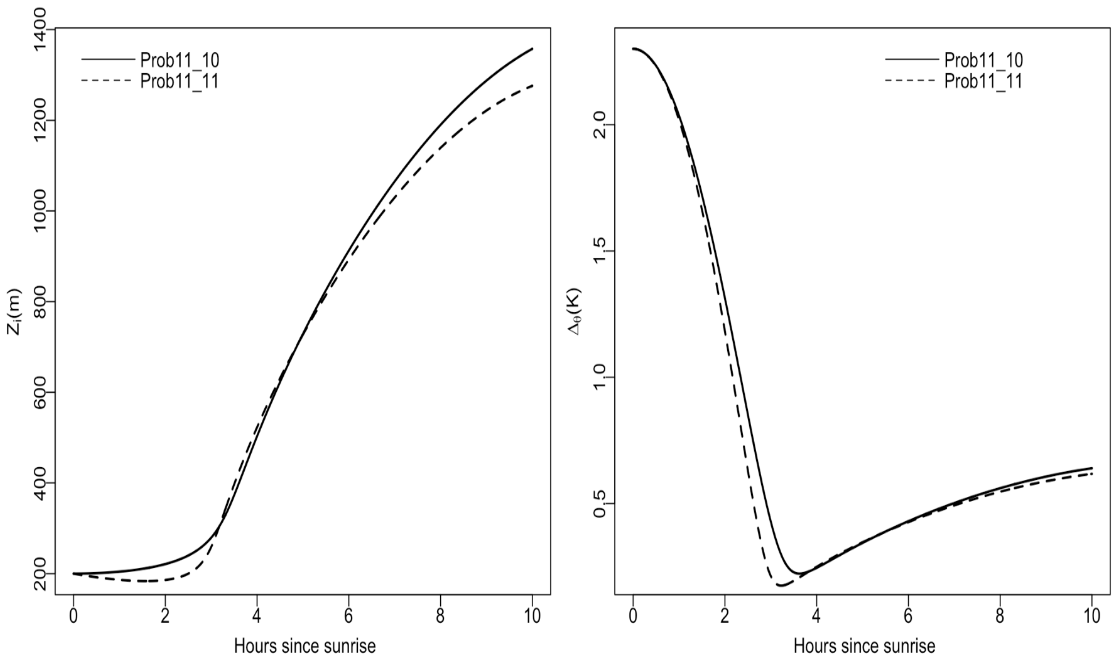

Figure 11.3: Comparison of the boundary layer height and the inversion temperature jump calculated from Problems 11.10 and 11.11

11.13 The surface CO_2 flux is given by

$$(\overline{w's_c'})_s(t) = -1.2\sin(t\pi/12) \qquad (0 < t < 12) \tag{11.3}$$

where $(\overline{w's_c'})_s$ is in $\mathrm{mg\,kg^{-1}\,m\,s^{-1}}$, and t is time in hours since sunrise. The initial CO_2 concentration jump across the capping inversion is -5 mg kg^{-1}. Using Original Equation 11.45 and the boundary height obtained in Problem 11.8, estimate the concentration jump Δ_c at hourly intervals from $t = 1$ h to 10 h.

Integrating Original Equation 11.45 with respect to time, we get:

$$\Delta_c = \frac{1}{z_i}\left[z_i(0)\Delta_c(0) + \frac{1.2 \times 12 \times 3600}{\pi}\left(1 - \cos\left(\frac{t\pi}{12}\right)\right)\right]$$

The estimated values for Δ_c is shown in Table 11.4

11.14 The entrainment heat flux is -0.03 K m s^{-1}, the temperature, water vapor and CO_2 inversion jumps are 1.6 K, -2.0 g kg^{-1}, and 12.3 mg kg^{-1}, respectively, in a convective boundary layer. Find the entrainment water vapor and CO_2 fluxes.

Use Original Equation 11.46 to get entrainment velocity:

$$w_e = -\frac{(\overline{w'\theta'})_{z_i}}{\Delta_\theta} = -\frac{-0.03 \text{ K m s}^{-1}}{1.6 \text{ K}} = 0.01875 \text{ m s}^{-1}$$

Table 11.4: CO_2 inversion jump for Problem 11.13.

time (hour)	Δ_c (mg kg^{-1})
1	-2.19
2	4.63
3	9.84
4	13.60
5	16.62
6	19.11
7	21.16
8	22.76
9	23.92
10	24.63

Entrainment carbon dioxide flux:

$$(\overline{w's'_c})_{z_i} = -w_e\Delta_c = -0.01875 \text{ m s}^{-1} \times 12.3 \text{ mg kg}^{-1} = -0.23 \text{ m s}^{-1} \text{ mg kg}^{-1}$$

Entrainment water vapor flux:

$$(\overline{w's'_v})_{z_i} = -\Delta_v w_e = -(-2.0 \text{ g kg}^{-1}) \times 0.01875 \text{ m s}^{-1} = 0.037 \text{ m s}^{-1} \text{ g kg}^{-1}$$

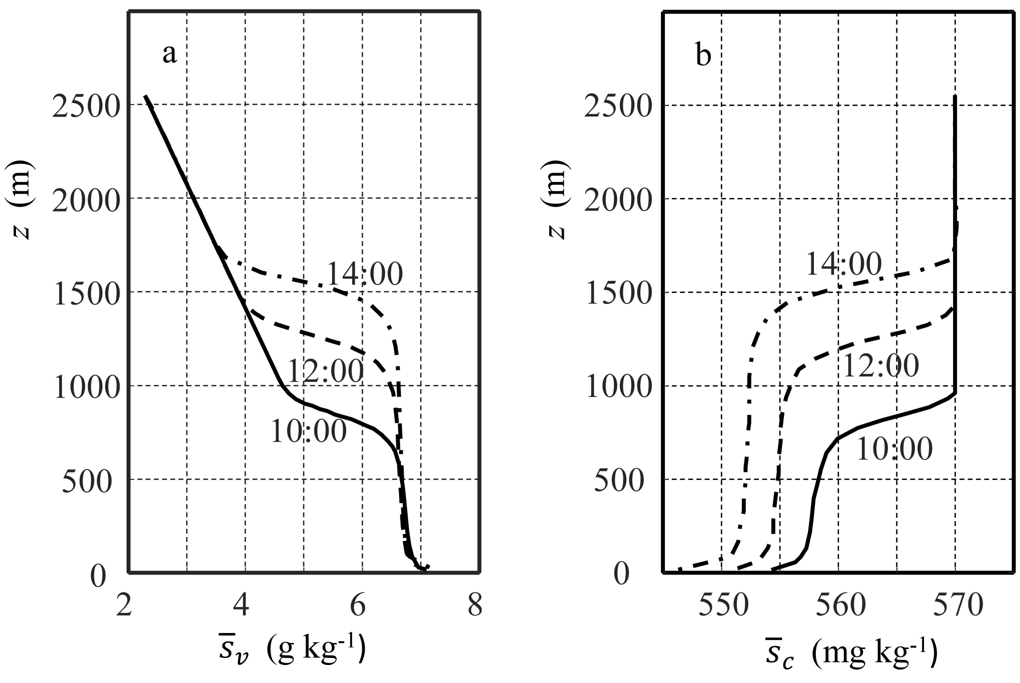

Figure 11.4: (Original Figure 11.13Profiles of water vapor (a) and carbon dioxide mixing ratio (b) in a cloud-free convective boundary layer. Time marks are local time. Data source: Huang et al. (2011).

11.15 Using the profile data shown in Figure 11.4a and the ABL technique (Equation 11.51), estimate the landscape water vapor flux and the latent heat flux between 10:00 and 14:00 local time. (Hint: there is no large-scale flow convergence or divergence.)

Since there is no large-scale flow convergence or divergence, $\overline{w} = 0$. The boundary layer height increases by roughly 700 m from 10:00 to 14:00 local time. The mean boundary layer water vapor mixing ratio does not change during this time. The mean boundary layer height (z_i) and the mean inversion jump (Δ_v) are 1150 m and -2.25 $\mathrm{g\,kg^{-1}}$. Plugging these into Original Equation 11.51, we get:

$$
\begin{aligned}
(\overline{w's'_v})_s &= z_i \frac{\partial \overline{s}_{v,m}}{\partial t} - \Delta_v\left(\frac{\partial z_i}{\partial t} - \overline{w}\right) \\
(\overline{w's'_v})_s &= -(\Delta_v)\left(\frac{\partial z_i}{\partial t}\right) \\
&= 0.11\ \mathrm{g\,kg^{-1}\ m\,s^{-1}}
\end{aligned}
$$

Assuming a dry air density of 1.2 kg $\mathrm{m^{-3}}$ and using the latent heat of vaporization of 2260 J $\mathrm{g^{-1}}$, the latent heat flux between 10:00 and 14:00 local time is $0.11 \times 1.2 \times 2260 = 298.4$ W $\mathrm{m^{-2}}$.

11.16 Using the profile data shown in Figure 11.4b and the ABL technique (Equation 11.47), estimate the surface carbon dioxide flux between 10:00 and 14:00 local time. (Hint: there is no large-scale flow convergence or divergence.)

Use Equation 11.47 to estimate the surface carbon dioxide flux:

$$(\overline{w's'_c})_s = z_i \frac{\partial \overline{s}_{c,m}}{\partial t} - \Delta_c\left(\frac{\partial z_i}{\partial t} - \overline{w}\right)$$

We can assume that $\overline{w}$ is 0 because there is no large-scale flow convergence nor divergence. We can estimate $\frac{\partial \overline{s}_{c,m}}{\partial t}$ and $\frac{\partial \overline{z_i}}{\partial t}$ using data shown in Original Figure 11.13.

At 10 am, $\overline{s}_{c,m}$ is 558 $\mathrm{mg\,kg^{-1}}$ and z_i is 1000 m. At 2 pm, $\overline{s}_{c,m}$ is 552 $\mathrm{mg\,kg^{-1}}$ and z_i is 1600 m. Therefore, the rates of change over 4 hours are -0.00042 $\mathrm{mg\,kg^{-1}\,s^{-1}}$ and 0.042 $\mathrm{m\,s^{-1}}$. The mean z_i and Δ_c between these two times are 1300 m and 15 $\mathrm{mg\,kg^{-1}}$. Plugging these values in Original Equation 11.47:

$$(\overline{w's'_c})_s = 1300 \times (-0.00042) - 15 \times 0.042 = 1.18\ \mathrm{mg\,kg^{-1}\,m\,s^{-1}}$$

If you multiply this value by dry air density of 1.20 $\mathrm{kg\,m^{-3}}$ you get 1.42 $\mathrm{mg\,m^{-2}\,s^{-1}}$ or 32.3 $\mu\mathrm{mol\,m^{-2}\,s^{-1}}$.

11.17 For an Obukhov length of 20.5 m, a surface friction velocity of 0.22 $\mathrm{m\,s^{-1}}$, find the boundary layer height.

Use Equation 11.53 to determine the boundary layer height using the Zilitinkevich relation (in stable conditions):

$$
\begin{aligned}
z_i &\simeq 0.4(u_* L/f)^{1/2} \\
&= 0.4(0.22\ \mathrm{m\,s^{-1}} \times 20.5\ \mathrm{m}/(1\times 10^{-4}\ \mathrm{s^{-1}}))^{1/2} \\
&= 84.95\ \mathrm{m}
\end{aligned}
$$

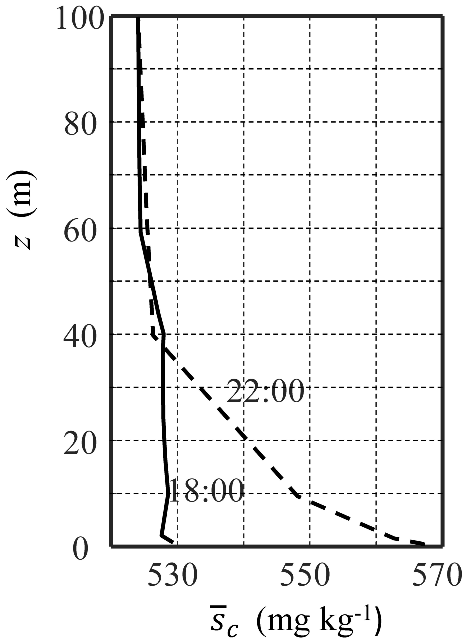

Figure 11.5: (Original Figure 11.14) Tethersonde profiles of CO_2 mixing ratio in a nocturnal boundary layer over a pasture land. Data source: Denmeand et al. (1996).

11.18 Using the profiles of CO_2 mixing ratio shown in Figure 11.5 and the nocturnal ABL technique (Original Equation 11.12), find the surface CO_2 flux in $\mu\mathrm{mol\,m^{-2}s^{-1}}$. Is the surface a source or sink of atmospheric CO_2?

We use Original Equation 11.52:

$$
z_i \frac{\partial \overline{s}_{y,m}}{\partial t} = (\overline{w's'_y})_s
$$

We know from the data shown in Figure 11.6 that:

$$
z_i = 40\ \mathrm{m}
$$

$$\frac{\partial \bar{s}_{y,m}}{\partial t} = \frac{540 \text{ mg}\,\text{kg}^{-1} - 530 \text{ mg}\ \text{kg}^{-1}}{22 - 18 \text{ h}} = 0.00069 \text{ mg}\,\text{kg}^{-1}\,\text{s}^{-1}$$

So the surface flux:

$$(\overline{w's'_y})_s = 40 \text{ m} \times 0.00069 \text{ mg}\,\text{kg}^{-1}\,\text{s}^{-1} = 0.028 \text{ m}\,\text{s}^{-1}\,\text{mg}\ \text{kg}^{-1}$$

Multiplying this value by dry air density of 1.2041 $\text{kg}\,\text{m}^{-3}$, the flux is 0.034 $\text{mg}\,\text{m}^{-2}\,\text{s}^{-1}$. Dividing this value by 0.044 $\text{kg}\,\text{mol}^{-1}$, we get 0.77 $\mu\text{mol}\,\text{m}^{-2}\,\text{s}^{-1}$.

The surface is a source of atmospheric CO_2.

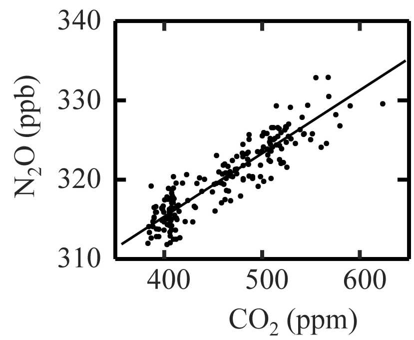

Figure 11.6: (Original Figure 11.15) Correlation between nitrous oxide and carbon dioxide molar mixing ratios observed in the stable boundary layer above a paddock grazed by sheep. The solid line represents the best-fit regression equation $y = 284 + 0.080x$. The surface CO_2 flux is 0.20 $\text{mg}\,\text{m}^{-2}\text{s}^{-1}$ during the measurement period. Data source: Kelliher et al. (2002).

11.19 Using the tracer correlation method and the data provided in Figure 11.6, find the surface nitrous oxide flux.

The relationship between two tracer gas surface fluxes is given by Original Equation 11.56. Using Original Figure 11.15, we can solve for the surface nitrous oxide flux:

$$(\overline{w's'_{N2O}})_s = (0.20 \text{ mg}\,\text{m}^{-2}\,\text{s}^{-1})(0.080) = 0.016\ \mu\text{g}\,\text{m}^{-2}\,\text{s}^{-1}$$

11.20 Using the equilibrium ABL technique and the data provided in Table 11.5, find the monthly mean landscape CO_2 flux.

Table 11.5: Monthly mean carbon dioxide and water vapor mixing ratios above ($\hat{s}_{c,+}$ and $\hat{s}_{v,+}$) and in the atmospheric boundary layer ($\hat{s}_{c,m}$ and $\hat{s}_{v,m}$), and monthly mean land surface vevapotranspiration rate (ET) in Wisconsin, USA. Data source: Helliker et al. (2004).

Month	$\hat{s}_{c,+}$ mg kg^{-1}	$\hat{s}_{c,m}$ mg kg^{-1}	$\hat{s}_{v,+}$ g kg^{-1}	$\hat{s}_{v,m}$ g kg^{-1}	ET mmol m^{-2}s^{-1}
Jan	564.2	571.0	0.8	1.7	0.1
Feb	564.9	571.6	0.9	2.8	0.1
Mar	566.0	573.3	1.0	3.5	0.3
Apr	568.0	571.3	1.1	3.1	0.6
May	567.2	565.1	1.6	7.2	1.4
Jun	562.5	554.6	1.9	8.7	2.0
Jul	556.4	542.5	2.0	10.8	2.4
Aug	552.0	543.5	1.8	10.8	2.0
Sep	551.7	555.4	1.7	7.4	1.5
Oct	556.6	564.2	1.1	5.1	0.6
Nov	562.4	570.2	1.0	3.2	0.2
Dec	565.5	569.8	0.7	1.5	0.1

In order to use Original Equation 11.60 for the equilibrium ABL technique, we must solve for W by inverting Original Equation 11.61:

$$W = \frac{\hat{F}_{v,s}}{\rho_d(\hat{s}_{v,+} - \hat{s}_{v,m})}$$

Solve mean monthly W values using the ET value for $\hat{F}_{v,s}$. For example, in Jan:

$$W = \frac{0.1\ \text{mmol}\ \text{m}^{-2}\,\text{s}^{-1}}{1.2041\ \text{kg}\,\text{m}^{-3}(0.8\ \text{g}\,\text{kg}^{-1} - 1.70\text{g}\,\text{kg}^{-1})} = -0.09\ \text{mmol}\,\text{m}\,\text{s}^{-1}\,\text{g}^{-1}$$

We then use Original Equation 11.60 to solve for mean monthly CO_2 flux:

$$\hat{F}_{c,s} = W\rho_d(\hat{s}_{c,+} - \hat{s}_{c,m})$$

For example, in Jan:

$$\begin{aligned}\hat{F}_{c,s} &= -0.09\ \text{mmol}\,\text{m}\,\text{s}^{-1}\,\text{g}^{-1} \times 1.2041\ \text{kg}\ \text{m}^{-3}(564.2\ \text{mg}\ \text{kg}^{-1} - 571\ \text{mg}\ \text{kg}^{-1})\\ &= 0.76\ \mu\text{mol}\,\text{m}^{-2}\,\text{s}^{-1}\end{aligned}$$

See the table below for the other mean monthly CO_2 fluxes.

11.21* The surface CO_2 flux is given by Equation 11.3, the initial CO_2 inversion jump Δ_c is -5 mg kg^{-1}, and the initial column mean CO_2 concentration $\overline{s}_{c,m}$ is 575 mg kg^{-1}. Other constraints are given in Problem 11.10. Solve numerically Δ_c, $\overline{s}_{c,m}$ and the entrainment CO_2 flux $(\overline{w's_c'})_{z_i}$ for $t = 1$ to 10 h.

Approach 1:
Steps for numerical solution (using a time interval of 1 second):

1. Calculate surface heat flux $(\overline{w'\theta'})_s$ from original Equation 11.63.

2. Calculate entrainment heat flux $(\overline{w'\theta'})_{z_i}$ using the result of step 1 and an entrainment ratio (A_T) of 0.2 in Original Equation 11.36.

Table 11.6: Monthly mean mixing velocity W (mmol m s^{-1} g^{-1}) and surface carbon dioxide flux $\hat{F}_{c,s}$ (µmol m^{-2} s^{-1}).

Month	W	$\hat{F}_{c,s}$
Jan	-0.09	0.76
Feb	-0.04	0.35
Mar	-0.10	0.88
Apr	-0.25	0.99
May	-0.21	-0.53
Jun	-0.24	-2.32
Jul	-0.23	-3.79
Aug	-0.18	-1.89
Sep	-0.22	0.97
Oct	-0.12	1.14
Nov	-0.08	0.71
Dec	-0.10	0.54

3. Calculate change in boundary layer height (z_i) and mean boundary layer potential temperature ($\overline{\theta}_m$) during one time step using Original Equations 11.21 and 11.32, respectively.

4. Calculate change in inversion strength (Δ_θ) during one time step using Original Equation 11.35.

5. Find the new z_i, $\overline{\theta}_m$, and Δ_θ for the next time step.

6. Find the surface carbon dioxide flux $(\overline{w's_c'})_s$ from Original Equation 11.64.

7. Find the carbon dioxide entrainment flux from Original Equation 11.41.

8. Find the change in mean carbon dioxide mixing ratio in the boundary layer and carbon dioxide inversion jump during one time step from Original Equations 11.40 and 11.43, respectively.

9. Find the new z_i, $\overline{\theta}_m$, Δ_θ, Δ_c, and $\bar{s}_{c,m}$ for the next time step.

10. Start again from step 1.

Figure 11.7 shows the change in carbon dioxide entrainment flux, mean carbon dioxide mixing ratio, and carbon dioxide inversion jump during ten hours after sunrise.

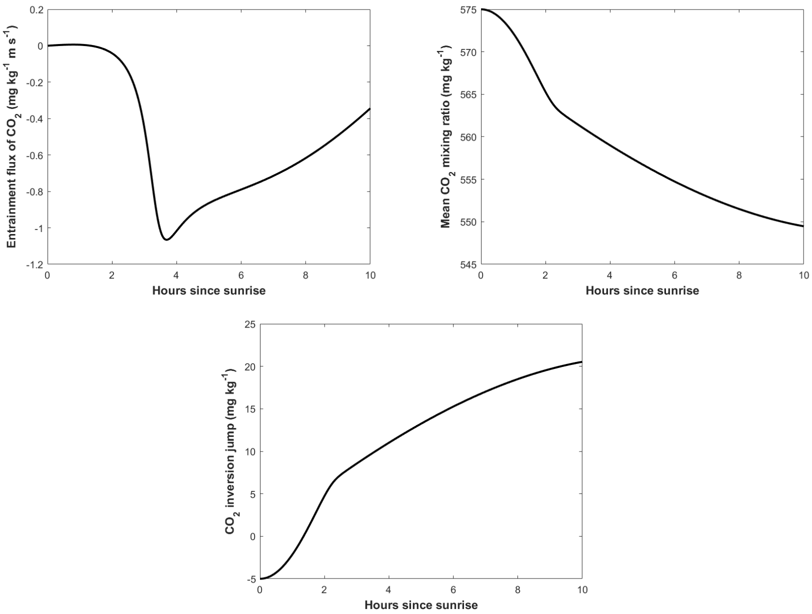

Figure 11.7: Change in carbon dioxide entrainment flux, mean carbon dioxide mixing ratio, and carbon dioxide inversion jump during the first ten hours after sunrise.

The MATLAB code is below:

```
%time steps
t=[1:1:36000];
%Initial boundary layer height
z=200;
%CO2 inversion jump
DelCO=-5;
%Potential temperature gradient in free atmosphere
lamb=.0033;
%Large-scale mean vertical velocity
w=0;
%Mean C02 mixing ratio of boundary layer
CO2m=575;
%Change in boundary layer height per time step
delz=0;
%Temperature inversion jump
Del=2.3;
%Change in temperature inversion jump per time step
delDel=0;
%Change in CO2 inversion jump per time step
delDelCO=0;
%Change in mean C02 mixing ratio of boundary layer per time step
delCO2m=0;
%Mean potential temperature of boundary layer
Thetam=281.1;
%Change in potential temperature of boundary layer per time step
delThetam=0;
for i=1:length(t)
    %Original Equation 11.63
    H=0.1*sin(3.14*(t(i)/3600)/12);
     %Original Equation 11.36
    Hz=-0.2*H;
    %Original Equation 11.32
    delThetam=(H-Hz)/z;
    %Original Equation 11.21
    delz=w-(Hz/Del);
```

```
     %Original Equation 11.35
    delDel=lamb*(delz-w)-delThetam;
    %Original Equation 11.64
    CO2=-1.2*sin(3.14*(t(i)/3600)/12);
    %Original Equation 11.41
    CO2z=-DelCO*(delz-w);
    %Original Equation 11.40
    delCO2m=(CO2-CO2z)/z;
    %Original Equation 11.43
    delDelCO=-delCO2m;
    %Boundary layer height for next time step
    z=z+delz;
    %Inversion jump for next time step
    Del=Del+delDel;
    %C02 inversion jump for next time step
    DelCO=DelCO+delDelCO;
    %Mean C02 mixing ratio of boundary layer for next time step
    CO2m=CO2m+delCO2m;
    %Mean potential temperature of boundary layer for next time step
    Thetam=Thetam+delThetam;
     %Create dataset for CO2 flux at top of boundary, mean CO2 mass mixing ratio,
    %and CO2 inversion jump
    EnCO2(i)=CO2z;
    CO2f(i)=CO2m;
    Inv(i)=DelCO;
end
```

Approach 2:
The entrainment CO_2 flux is given by Original Equation 11.41:

$$(\overline{w's_c'})_{z_i} = -\Delta_c \frac{\partial z_i}{\partial t}$$

Inserting the above equation and Original Equation 11.64 into Original Equation 11.40, we obtain:

$$\frac{\partial \overline{s}_{c,m}}{\partial t} = \frac{1}{z_i}\left(\Delta_c \frac{\partial z_i}{\partial t} - 1.2\sin\left(\frac{\pi t}{12 \times 3600}\right)\right)$$

The CO_2 inversion jump Δ_c obeys:

$$\frac{\partial \Delta_c}{\partial t} = -\frac{\partial \overline{s}_{c,m}}{\partial t}$$

Combining the above three equations with the two equations in Problem 11.10, we can obtain the numerical solution of z_i,Δ_θ, $(\overline{w's_c'})_{z_i}$, Δ_c and $\overline{s}_{c,m}$. The results are displayed in the figure below.

```
# R code:
#Prob. 11.21: mean vertical velocity(w) is zero;
#set model parameters
pars <- c(r=3.3/10^3,#gamma  K/km^3
         )
#set initial values
yini <- c(Z = 200,
          D = 2.3,
         C=-5,# initial CO2 inversion jump(Δ_c) mg/kg
          S=575,# initial column mean CO2 concentration(s_cm) mg/kg
          )
times <- seq(0, 10*3600, by = 60)
#set differential equations
LVmod0D <- function(t, State, Pars){
 with(as.list(c(t,State, Pars)),{
    dZ <- 0.02*sin(pi/12*t/3600)/D
```

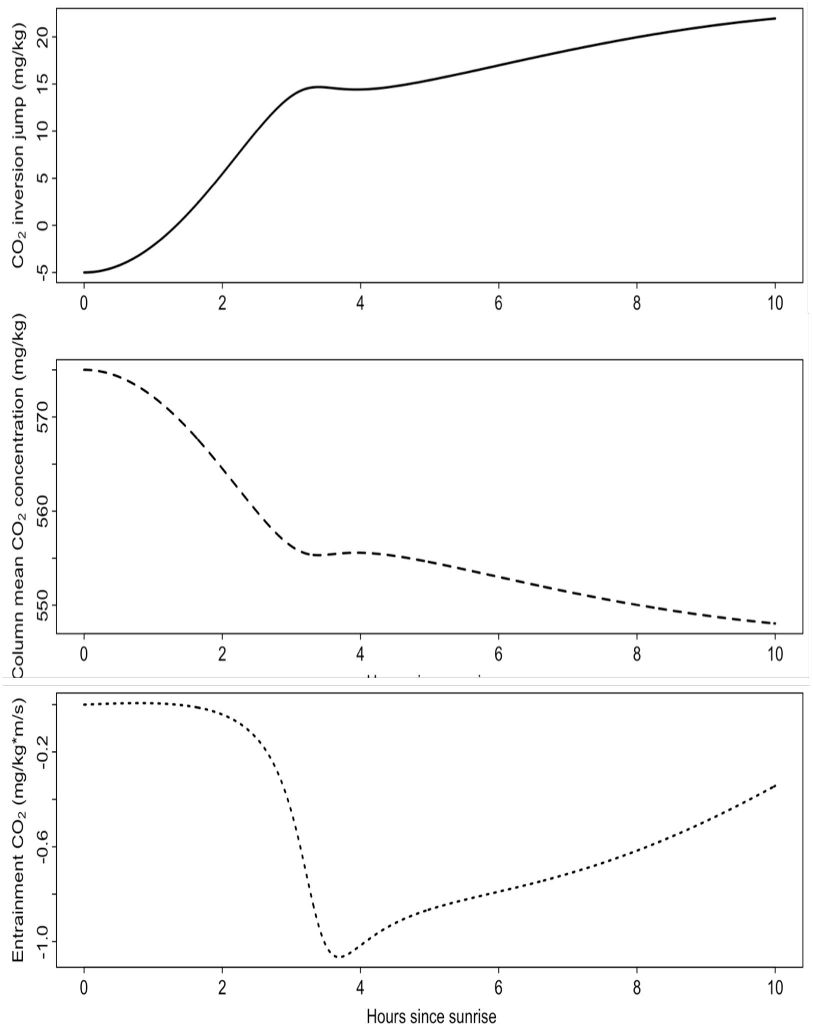
CO2 inversion jump (mg/kg)
-5
0
5
10
15
20
Column mean CO2 concentration (mg/kg)
550
560
570
Entrainment CO2 (mg/kg*m/s)
-1.0
-0.6
-0.2
0
2
4
6
8
10
Hours since sunrise

Figure 11.8

```
    dD <- 0.1*(0.2*r/D-1.2/Z)*sin(pi/12*t/3600)
    dS<-C*dZ/Z-1.2*sin(pi/12*t/3600)/Z
    dC<-(-dS)
    return(list(c(dZ,dD,dS,dC)))
  })
}
out3<-ode(func=LVmod0D,y=yini,parms=pars,times=times)
H<-(-out3[,4]*0.02*sin(pi/12*out3[,1]/3600)/out3[,3]) # entrainment CO2 flux ...
    mg/kg*m/s
#out3
```

11.22 Derive Original Equation 11.41 from the CO_2 conservation equation for the capping inversion layer using the same strategy that yields Original Equation 11.21.

We start with the conservation equation for the capping inversion layer:

$$\frac{\partial \overline{s}_c}{\partial t} + w\frac{\partial \overline{s}_c}{\partial z} = -\frac{\partial \overline{w' s_c'}}{\partial z}$$

Integrating the time change term, we have:

$$\int_{z_i-\varepsilon}^{z_i+\varepsilon} \frac{\partial \overline{s}_c}{\partial t} dz = \frac{\partial}{\partial t}\int_{z_i-\varepsilon}^{z_i+\varepsilon} \overline{s}_c dz - \left[\overline{s}_c\left(z_i+\varepsilon\right) - \overline{s}_c\left(z_i-\varepsilon\right)\right]\frac{\partial z_i}{\partial t}$$

where we have used the Leibniz integral rule for differentiation:

$$\frac{\partial}{\partial t}\int_{x_1(t)}^{x_2(t)} y\left(z,t\right)dz = \int_{x_1(t)}^{x_2(t)} \frac{\partial y}{\partial t}dz + y\left(x_2\left(t\right),t\right)\frac{\partial x_2}{\partial t} - y\left(x_1\left(t\right),t\right)\frac{\partial x_1}{\partial t}$$

We note that:

$$\overline{s}_c\left(z_i-\varepsilon\right) = \overline{s}_{c,m}$$

In the limit $\varepsilon \to 0$:

$$\int_{z_i-\varepsilon}^{z_i+\varepsilon} \overline{s}_c dz \to 0,$$

and

$$\overline{s}_c\left(z_i+\varepsilon\right) \to \overline{s}_{c,+}$$

So we obtain:

$$\lim_{\varepsilon\to 0}\int_{z_i-\varepsilon}^{z_i+\varepsilon} \frac{\partial \overline{s}_c}{\partial t}dz = -\Delta_c\frac{\partial z_i}{\partial t}$$

and we also have:

$$\int_{z_i-\varepsilon}^{z_i+\varepsilon} \overline{w}\frac{\partial \overline{s}_c}{\partial z}dz = \left[\overline{s}_c\left(z_i+\varepsilon\right) - \overline{s}_c\left(z_i-\varepsilon\right)\right]\overline{w}$$

The limit of this integral is:

$$\lim_{\varepsilon\to 0}\int_{z_i-\varepsilon}^{z_i+\varepsilon} \overline{w}\frac{\partial \overline{s}_c}{\partial t}dz = -\Delta_c\overline{w}$$

$$\lim_{\varepsilon\to 0}\left\{-\int_{z_i-\varepsilon}^{z_i+\varepsilon} \frac{\partial \overline{w' s_c'}}{\partial z}dz\right\} = \left(\overline{w' s_c'}\right)_{z_i}$$

Combining the above equations, we obtain a prognostic equation for the entrainment CO_2 flux:

$$\left(\overline{w' s'_c}\right)_{z_i} = -\Delta_c\left(\frac{\partial z_i}{\partial t} - \overline{w}\right)$$

11.23 Derive Equation 11.41 using the same method deployed for the derivation of Equation 11.21.

The CO_2 mass mixing ratio at the bottom of the free atmosphere can be expressed as:

$$\overline{s}_{c,+} = \gamma_c \left(z_i - z_f\right) + \overline{s}_c \left(z_f\right)$$

Differentiating the equation above with respect to time and making use of Original Equation 11.6, we obtain:

$$\frac{\partial \overline{s}_{c,+}}{\partial t} = \gamma_c \frac{\partial z_i}{\partial t} - \gamma_c \overline{w}$$

Making use of:

$$\Delta_c = \overline{s}_{c,+} - \overline{s}_{c,m}$$

we obtain the prognostic equation for Δ_c:

$$\frac{\partial \Delta_c}{\partial t} = \gamma_c \left(\frac{\partial z_i}{\partial t} - \overline{w}\right) - \frac{\partial \overline{s}_{c,m}}{\partial t}$$

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